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SPHERICAL TRIGONOMETRY

THEORETICAL AND PRACTICAL

EXEMPLIFIED BY
THE SOLUTION OF A LARGE NUMBER OF
SPHERICAL TRIANGLES

ADAPTED FOR THE USE OF STUDENTS PREPARING FOR THE
FOLLOWING EXAMINATIONS
B.A., LONDON; LIEUTENANT, R.N.;
ROYAL MILITARY ACADEMY, INTERMEDIATE;
DEPARTMENT OF SCIENCE AND ART.
MATHEMATICS, FOURTH STAGE

BY

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PREFACE

THE following treatise contains demonstrations and practical illustrations of the most important rules used in the solution of Spherical Triangles.

The principles of Solid Geometry, on which the propositions of Spherical Geometry and Trigonometry depend, are so few in number and so simple that a student who is unacquainted with the eleventh book of Euclid may easily convince himself of their truth by the aid of models. These need not be specially made, they are present everywhere about us: the sides and bottom of a box, an open book, the corner of a box, for example, will respectively illustrate planes perpendicular to a plane, angles between two planes, solid angles contained by three plane angles, and so on. Illustrative models may also be readily made by folding a sheet of paper.

These principles being understood, the propositions of Spherical Geometry are then proved as far as possible directly from them.

Each proposition is demonstrated separately and illustrated where necessary by its own figure.

In Chapter vi. four geometrical proofs will be found, three of which it is believed appear for the first time. These are, however, only verifications for particular cases, but they could be easily extended by using the figure of § 29.

The rule of sines has been dealt with exhaustively with a view to making the explanations quite clear.

For the practical solution of spherical triangles no simpler or more concise rules can probably be given than those demonstrated in Chapter vii. and practically illustrated in Chapter xi. They involve, however, the use of the L haversine and tabular versed sine tables. These are unfortunately not included in the tables allowed to be used at examinations conducted by the Civil Service

Commissioners, and accordingly other rules have been established, and the student will find that he has ample choice.

Entirely new and very numerous figures have been provided with the view of getting the eye to help the reasoning. Those which illustrate properties of circles of the sphere and of spherical triangles have been drawn in perspective. For the more practical work, however, stereographic projection has been made use of, as by this means a simplification results from the fact that great circles passing through the observer's eye are projected into straight lines passing through the centre of projection, and all other great circles appear as circles intersecting at the same angles as they actually do on the sphere.

Students are recommended to solve the spherical triangles approximately at first, taking the tabular log ratios given in the tables they are accustomed to. This they should continue to do until they are perfectly familiar with the method of the rules. Proportioning to seconds is tedious and quite unnecessary for ordinary work.

To Mr. J. Humphrey Spanton, Drawing Instructor to the Cadets of H.M.S. *Britannia*, the author would express his hearty thanks for kind assistance with the figures, and to the Rev. J. C. P. Aldous, M.A., Chief Instructor of H.M.S. *Britannia*, he is greatly obliged for much helpful advice.

He would also express his general indebtedness for ideas to the works of Robertson, Cape, Snowball, Todhunter, M'Clelland and Preston, Serret, Lacroix, Lefébure de Fourcy, and de Comberousse.

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SPHERICAL TRIGONOMETRY

CHAPTER I

GEOMETRICAL RELATIONS BETWEEN CIRCLES OF THE SPHERE

1. *Definition.*—**A sphere** is a solid bounded by one surface, every point of which is equally distant from a certain fixed point within the solid called the **centre** of the sphere.

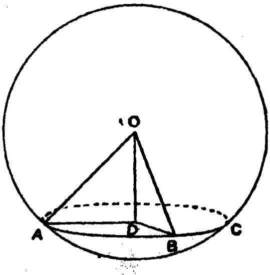
Def.—**A radius** of a sphere is a straight line drawn from the centre to any point in the surface.

Def.—**A diameter** of a sphere is a straight line drawn through the centre, terminated both ways by the surface.

Def.—A straight line is **perpendicular** to a plane when it makes right angles with all straight lines drawn to meet it in that plane (Euclid xi. def. 3).

2. *Any section of a sphere by a plane is a circle.*

Let ABC be a section of a sphere made by a plane. From O, the centre of the sphere, draw OD perpendicular to the plane.



In the boundary of the section take any points A, B.

Join OA, OB, DA, DB.

Since OD is perpendicular to the plane, therefore ODA, ODB are right angles.

Hence the squares on OD, DA are equal to the square on OA, also the squares on OD, DB are equal to the square on OB.

But the square on OA is equal to the square on OB.

Therefore the squares on OD, DA equal the squares on OD, DB.

Take away the common square on OD, and the remainder, the square on DA, is equal to the remainder, the square on DB.

And therefore the straight line DA is equal to the straight line DB.

But A and B are *any* points in the boundary ABC.

Therefore *all* points in the boundary ABC are equally distant from the point D.

Therefore ABC is a circle, and D is its centre.

3. *Def.*—**A great circle** is a section of a sphere made by a plane passing through the centre of the sphere.

Def.—**A small circle** is a section of a sphere made by a plane which does not pass through the centre of the sphere.

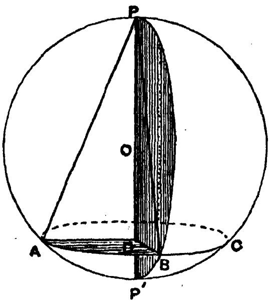
Def.—**The axis** of any circle of a sphere is that diameter of the sphere which is perpendicular to the plane of the circle.

Def.—**The poles** of a circle of a sphere are the extremities of its axis.

4. *A pole of a circle is equidistant from every point in the circumference of the circle.*

Let ABC be a circle of the sphere.

From O the centre of the sphere draw OD perpendicular to the plane of the circle, meeting it at D its centre (§ 2).



Produce OD to meet the surface of the sphere at P, P′.

Then POP′ is axis of the circle ABC; P and P′ are its poles.

In the circumference of the circle ABC take any points A, B.

Join PA, PB, DA, DB.

Then because AD equals DB (§ 2), and DP is common, also the angle PDA equals the angle PDB, each being a right angle.

Therefore PA equals PB (Euclid i. 4).

And all great circles of the same sphere are equal, since they have the radius of the sphere for their radii.

Therefore the arc PA equals the arc PB (Euclid iii. 28).

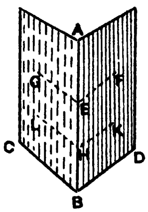
Also since OA equals OB and OP is common, and AP and BP are equal, therefore the angle POA equals the angle POB (Euclid i. 8).

But A and B are *any* points on the circumference of the circle ABC.

Therefore what we have proved for these *two* points must be true for all such points.

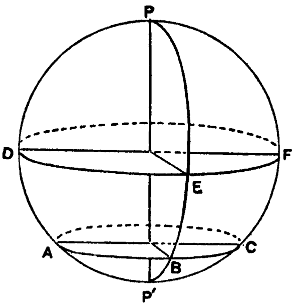
Hence a pole of a circle is equally distant from every point in the circumference of the circle whether we measure the distance by the length of a straight line joining the pole with the point, or by the arc of a great circle joining them, or by the angle which such an arc subtends at the centre of the sphere.

5. *Def.*—**The inclination of a plane to a plane** is the angle contained by two straight lines drawn from ~~any~~ the same point of their common section at right angles to it, one in one plane and the other in the other plane.



From points E, H, in AB the common section of the planes AC, AD, straight lines EG, HL are drawn in the plane AC perpendicular to AB, and also EF, HK perpendicular to AB in the plane AD. Then GEF, LHK are each of them the inclination of the planes AC, AD.

6. **Secondaries.**—Great circles which pass through the poles of a circle of the sphere are called *secondaries* to that circle.



Thus PEBP′ is a secondary to the small circle ABC, and to the great circle DEF.

7. Through *three* points not in the same straight line only *one* plane can be drawn, but through three points which are in the same straight line an infinite number of planes can be drawn (Euclid xi. 2).

8. *Through two points on the surface of a sphere only one great circle can be drawn except when those points are the extremities of the same diameter of the sphere, and then the number of great circles which can be drawn through them is infinite.*

Let A, B be two points on the surface of a sphere, not at extremities of the same diameter.

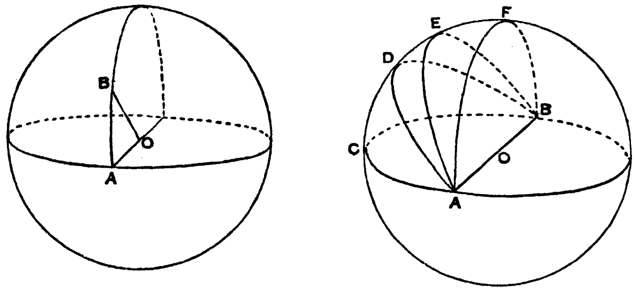
Take O, the centre of the sphere.

Join OA, OB, AB.

These three straight lines which meet one another lie in one plane (Euclid xi. 2), which passes through O the centre of the

sphere and cuts the surface of the sphere in the circumference of one great circle passing through A and B.

But when the points A, B are extremities of the same diameter, then A, B, and O, the centre of the sphere, are in the same

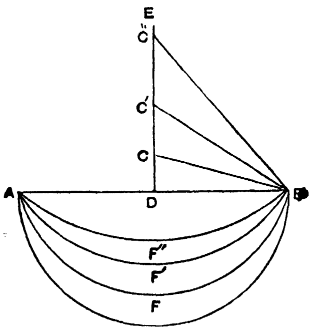


straight line, and therefore the number of planes through AOB is infinite.

And these planes will cut the sphere in an infinite number of great circles ACB, ADB . . . through A and B.

9. *Note.*—This proposition is of importance in Navigation, for it shows us that there can be only one great circle track between two places on the surface of the earth.

It is further of importance to note that this great circle arc is the shortest distance on the earth's surface between the two points. This may be made clear from a consideration of the figure.



A and B are two points joined by the straight line AB.

Bisect AB at right angles by DE.

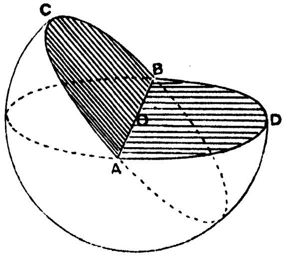
Then the centres of all circles passing through A, B will lie on DE.

As the centre moves along DE to C, C′, C″, . . . the radius CB, C′B, C″B . . . continually increases, and the curvature of the arcs AFB, AF′B, AF″B . . . continually decreases, till when the centre is at an infinite distance from D along DE the curve becomes the straight line AB or absolutely the shortest distance.

Hence since no greater radius can be taken for a circle of the sphere than the radius of the sphere, it follows that the great circle arc is the arc of least curvature, and hence the distance is least if measured along the arc of the great circle joining the points.

10. *Two great circles bisect one another at their points of intersection.*

Let ACB, ADB be two great circles intersecting at A and B. Then since ACB, ADB are *great* circles, their planes pass through O the centre of the sphere.



But their planes also pass through the points A and B.

Consequently the three points A, O, B are in the straight line which is the common section of their planes, and this straight line is a diameter of the sphere, and a common diameter of the great circles ACB, ADB.

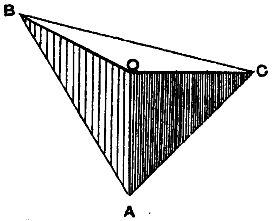
Hence ACB, ADB are semicircles.

Note.—This proposition may be otherwise stated as follows:

Two circles of the sphere cannot bisect one another unless their planes pass through the centre of the sphere (*i.e.* unless they are great circles). It is then the analogue of Euclid iii. 4 and may be proved by the aid of § 2, which is the analogue of Euclid iii. 3.

11. *Def.*—A **solid angle** is that which is made by more than two plane angles which are not in the same plane, meeting at one point.

The following case of a solid angle contained by three plane angles AOB, AOC, BOC deserves great attention, as it often occurs in propositions of spherical geometry.



If two of the angles AOB, AOC are right angles we have at once the following results:

(1) AO is perpendicular to the plane BOC (Euclid xi. 4).

(2) All planes through AO, *e.g.* the planes AOB, AOC, are perpendicular to the plane BOC (Euclid xi. 18).

(3) Since OA is the common section of the planes AOB, AOC, therefore BOC is the inclination of these planes (Euclid xi. def. 5).

Spherical Angle

12. *Def.*—A **spherical angle** is the inclination of two arcs of great circles at their point of intersection on the surface of a sphere.

Rectilineal Equivalents of a Spherical Angle

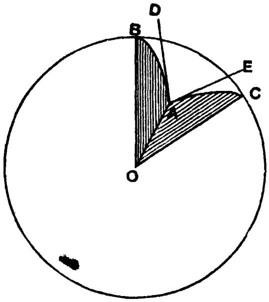
A spherical angle is equal to the following rectilineal angles:

(1) *The inclination of the tangents to the arcs at their point of intersection.*

In the figure, if AD, AE are respectively tangents to the arcs AB, AC, the rectilineal angle DAE is equal to the spherical angle BAC.

(2) *The angle between the planes of the great circles which form the spherical angle.*

In the figure, DAE is the angle between the planes AOB, AOC and at the same time it is equal to the spherical angle BAC.



13. Let BAC be a spherical angle.

Take O the centre of the sphere and join OA, OB, OC.

Draw tangents AD, AE to the arcs AB, AC respectively.

Then DAE equals the spherical angle BAC.

And because AD, AE are tangents to AB, AC respectively,

and OA is drawn from their common centre to the point of contact A,

therefore OAD, OAE are right angles.

But OA is the common section of the planes AOB, AOC.

Therefore DAE is the angle between these planes (§ 5).

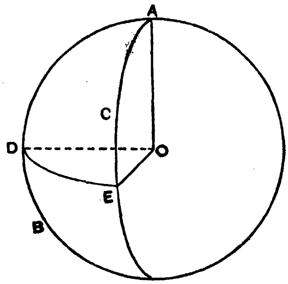
That is the spherical angle BAC is equal to the angle between the two planes of the great circles AB and AC.

Measures of a Spherical Angle

14. (1) *A spherical angle is measured by the arc of a great circle, which the containing arcs intercept on the great circle to which they are secondaries.*

Let BAC be a spherical angle. On AB, AC, produced if necessary, take AD, AE quadrants.

Take O the centre of the sphere, join OA, OD, OE.



Because AD, AE are quadrants and O is their common centre, therefore AOD, AOE are right angles.

But AO is the common section of the planes AOD, AOE,

therefore the angle DOE is the angle between these planes (§ 5).

But the spherical angle DAE is also the angle between these planes (§ 13).

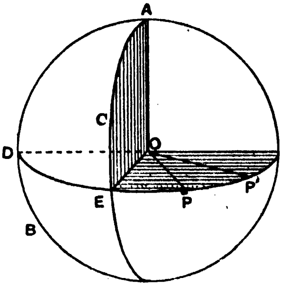
Therefore DOE is equal to the spherical angle DAE, but O is the centre of DE.

Therefore the arc DE measures the angle DOE, hence also the arc DE measures the spherical angle BAC.

15. (2) *A spherical angle is also measured by the arc of a great circle joining the poles of the great circles which contain the spherical angle.*

Let BAC be a spherical angle.

Let P be pole of AB and P′ pole of AC.



Join P, P′ by an arc of a great circle and produce it to meet AB and AC produced if necessary at D and E respectively.

Take O the centre of the sphere.

Join OA, OD, OE, OP, OP′.

Then because P is pole of AB, OP is axis of AB,
and therefore POA, POD, are right angles.

And because P′ is pole of AC, OP′ is axis of AC,
and therefore P′OA, P′OE are right angles.

Hence OA is perpendicular to OP, OP′ at their point of intersection O,
and therefore makes right angles with OD, OE which meet it in the plane POP′ (Euclid xi. 4).

But OA is the common section of the planes AOD, AOE,
therefore DOE is the angle between these planes, and consequently equals the spherical angle BAC (§ 13).

Now POD equals P′OE, each being a right angle.

Take away the common angle POE from each of these equals, and we have POP′ equals DOE.

But the arc PP′ measures POP′, since O is centre of PP′,
therefore also the arc PP′ measures the spherical angle BAC.

16. *Note.*—**Measure.** The relations between magnitudes of the *same* kind are *numerical* only. Hence the relation between a magnitude to be measured and a unit, which must be an invariable magnitude of the same kind, must be numerical only. In this sense the *measure* of a quantity is the number of times a unit is contained in it, and this measure will obviously be a *pure* number.

But when we say that an *arc measures* an *angle* the meaning is quite different.

One variable quantity is said to measure another of a *different kind* if it increases or diminishes indefinitely in the *same proportion* with it.

Thus if a line OB revolve about one extremity O from the initial position OA, the line generates an angle AOB whilst the point B generates the arc AB: and as the angle increases continuously so also does the arc in the *same* proportion. The arc AB is then said to be a *measure* of the angle AOB.

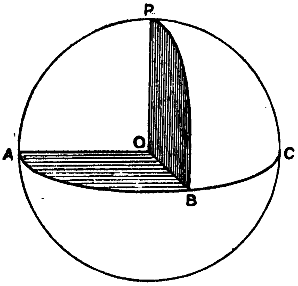
It is in this sense that the spherical angle BAC is *measured* by arc BC intercepted by AB. AC on the great circle to which AB, are secondaries.

Properties of the Pole of a Circle

17. *The arc of a great circle joining the pole of a great circle to any point in its circumference is a quadrant and is at right angles to that circle.*

Let P be a pole of the great circle ABC.

Then PA, PB are quadrants and are at right angles to ABC.



Take O the centre of the sphere.

Join OA, OB, OP.

Then because P is pole of ABC, OP is axis of ABC.

Therefore POA, POB are right angles.

But O is the common centre of the arcs PA, PB.

Therefore PA, PB are quadrants.

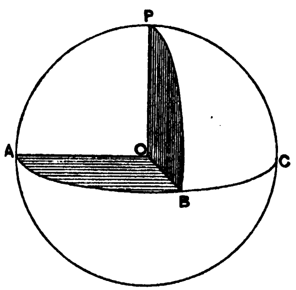
Again, because OP is perpendicular to the plane AOB, all planes through OP, *e.g.* the planes POA, POB, will be perpendicular to the third plane AOB (§ 11).

Hence the arcs PA, PB will be perpendicular to the arc ABC.

18. *If the arcs of great circles joining a point on the surface of a sphere with two other points on the surface of the sphere, which are not at opposite extremities of the same diameter, be each of them quadrants, then the first point is a pole of the great circle through the other two points.*

Let A and B be two points on the surface of a sphere, not at extremities of the same diameter.

Let P be a third point such that PA, PB are quadrants.



Then P shall be pole of the great circle AB.

From O the centre of the sphere draw OA, OB, OP.

Then because PA, PB are quadrants, and O is their common centre therefore POA, POB are right angles.

Hence OP is axis of AB, and P is a pole of AB (§ 3).

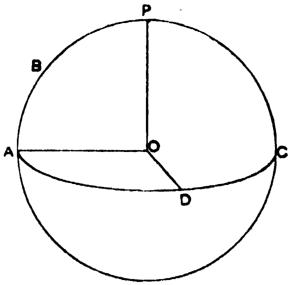
19. *If two great circles cut at right angles, each passes through the pole of the other.*

Let the great circles AB, AC make BAC a right angle.

On AB, AC produced if necessary take AP, AD quadrants.

Take O the centre of the sphere, and join OA, OD, OP.

Then because AP, AD are quadrants, and O is their common centre,



therefore POA, DOA are right angles,
but OA is the common section of the two planes POA, AOD,
therefore POD is the inclination of these planes (§ 5), and consequently equals the spherical angle PAD (§ 13), which by hypothesis is a right angle.

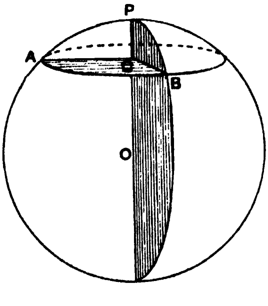
Now because POA, POD are right angles, PO is axis of AD, and P is a pole of AD (§ 3).

Similarly, because DOA, DOP are right angles, DO is axis of AB, and D is a pole of AB (§ 3).

20. If two planes which cut one another be each of them perpendicular to a third plane, their common section will be also perpendicular to the third plane (Euclid xi. 19).

21. *If from a point on the surface of a sphere there can be drawn two arcs of great circles not parts of the same great circle, the planes of which are at right angles to the plane of a given circle, that point is a pole of the circle.*

For since the planes of the great circles PA, PB are at right angles to the plane of AB, their line of intersection PCO is at right angles to the plane ACB.



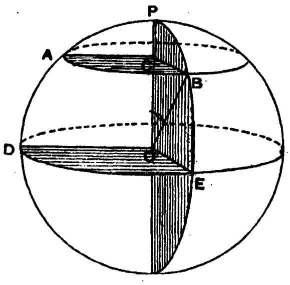
And at the same time CP passes through O the centre of the sphere,
therefore PCO is axis of AB and P is a pole of AB.

22. *To compare the arc of a small circle with the arc of a great circle intercepted between two great circles which pass through the common pole of both the circles.*

Let P be a pole of the small circle AB and of the great circle DE.

Draw the secondaries PAD, PBE.

Take O the centre of the sphere and join OP, cutting the plane of the small circle in C its centre (§ 2).



Join CA, CB, OD, OE, OB.

Then PCO is axis of AB and of DE, therefore PCA, PCB, POD, POE are right angles.

But PCO is the common section of the planes POD, POE,

therefore ACB and DOE are each of them the inclination of these planes (§ 5), that is ACB equals DOE.

Hence the circular measure of ACB equals the circular measure of DOE,

i.e. $$\frac{\text{arc AB}}{\text{BC}} = \frac{\text{arc DE}}{\text{OE}},$$

$$\therefore \quad \frac{\text{arc AB}}{\text{arc DE}} = \frac{\text{BC}}{\text{OE}} = \frac{\text{BC}}{\text{OB}} = \sin \text{BOC} = \cos \text{BOE} = \cos \text{BE}.$$

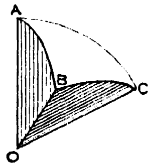
Note.—In Navigation the formulæ of Parallel sailing and of Mid. lat. sailing are derived from this proposition.

CHAPTER II

SPHERICAL TRIANGLES

23. **Spherical Triangle.**—If the angular point of a solid angle contained by three plane angles be made the centre of a sphere, the plane faces will cut the surface of the sphere in three arcs of great circles. These arcs by their intersection form a figure which is called a *Spherical Triangle.*

24. Suppose a solid angle contained by three plane angles, AOB, AOC, BOC to have its angular point O at the centre of a sphere.



Then the planes AOB, AOC, BOC will cut out on the surface of the sphere three arcs of great circles, AB, AC, BC forming a spherical triangle ABC.

The arcs AB, AC, BC are called **sides** of the spherical triangle.

The angles formed by these arcs at the points where they meet, namely BAC, ABC, ACB, are called the **angles** of the spherical triangle.

The angles are denoted by A, B, C, and the sides respectively opposite these angles by a, b, c.

25. Since the sides AB, AC, BC are arcs of great circles, therefore O, the centre of the sphere, is their common centre, and consequently the arcs AB, AC, BC respectively measure the angles AOB, AOC, BOC, that is the angles of the plane faces.

26. The angles of the spherical triangle are the same as the angles *at* which the plane faces are inclined (§ 13). Thus the spherical angle BAC is the angle between the two planes AOB, AOC, the angle ABC is the angle between the two planes AOB, BOC, and the angle ACB is the angle between the two planes AOC, BOC (§ 13).

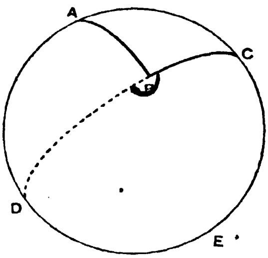
Limits for Angles and Sides of a Spherical Triangle

27. Since Euclid takes two right angles as the limit of a plane angle, this must also be the limit for any angle of a plane face of a solid angle.

Hence in a spherical triangle no side can be as great as a semicircle (that is, in sexagesimal measure, 180°).

28. *An angle of a spherical triangle must be less than two right angles.*

For, if possible, let ADECB be a spherical triangle having the angle ABC greater than two right angles.



Produce one of the containing sides CB to meet the third side at D.

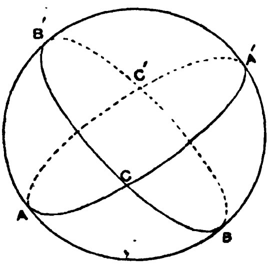
Then CED is a semicircle, and therefore CEDA is greater than a semicircle, which is impossible.

CHAPTER III

GEOMETRICAL RELATIONS BETWEEN THE SIDES AND ANGLES OF SPHERICAL TRIANGLES

29. **Polar Triangles.**—Since all great circles bisect each other, it will follow that three great circles by their intersection will divide the surface of the sphere into *four pairs of equal triangles.*

Thus in the figure since ACA′, CA′C′ are semicircles they are equal.



Take from each the common arc A′C, and we have

$$AC = A'C'.$$

Similarly $\quad AB = A'B'$

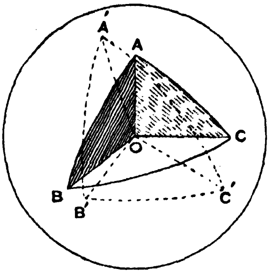
and $\quad BC = B'C'.$

Hence the triangle		ABC is equal to the triangle			A′B′C′.
Similarly	,,	A′BC	,,	,,	AB′C′,
,,	,,	A′B′C	,,	,,	ABC′,
,,	,,	AB′C	,,	,,	A′BC′.

30. Every great circle has two poles, and the three great circles which join the six poles of three great circles intersect and form four pairs of equal triangles. Each one of these triangles is connected with some one of the former group of triangles by the relation that the sides and angles of the one are respectively supplements of the angles and sides of the other.

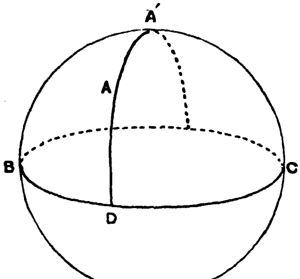
Such pairs of triangles are hence often called *Supplemental Triangles.*

31. *Def.*—**Polar Triangle.** Let ABC be a spherical triangle, and let A′, B′, C′ be those poles of the arcs BC, AC, AB respectively which lie on the same sides of those arcs as the opposite



angles A, B, C; then the triangle A′B′C′ is said to be the *Polar Triangle* of the triangle ABC. And the triangle ABC is called the *Primitive Triangle* with respect to the triangle A′B′C′.

32. If two points, one of which is pole of a great circle, be on the *same* side of that circle, the arc of a great circle which joins them is *less* than a quadrant, and conversely if the arc of a great circle joining two points, one of which is pole of a great circle, be less than a quadrant, the two points are on the *same* side of that circle.



Let A, A′ be two points on the *same* side of the great circle BDC, and let A′ be pole of BDC.

Then AA′ shall be less than a quadrant.

For draw the secondary A′AD through A, then A′D is a quadrant, and therefore A′A is less than a quadrant.

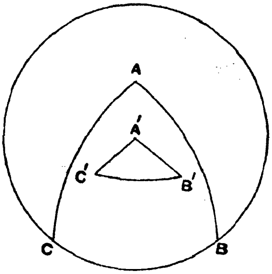
Also if A′A be less than a quadrant and A′ pole of BDC, then A′D is a quadrant; hence A and A′ are on the *same* side of BDC.

33. Some propositions with rather complicated figures become clearer by projecting them on the plane of one of the great circles involved. For the pole of the circle on whose plane the figure is described becomes the centre of that circle, and all secondaries to it appear as straight lines through this centre.

Example. The polar triangle of a given spherical triangle may be conveniently described on the plane of one of the sides of the primitive triangle.

Let the figure be described on the plane of BC, one of the sides of the triangle ABC.

Let A′, B′, C′ be those poles of the arcs BC, AC, AB respectively which lie on the same sides of those arcs as the opposite angles A, B, C.



Then A′B′C′ is a projection of the polar triangle to ABC on the plane of BC.

A′ will appear as the centre of BC, and the projections of the arcs A′B′, A′C′ will be straight lines.

34. *If one triangle be the polar triangle of another, the latter will be the polar triangle of the former.*

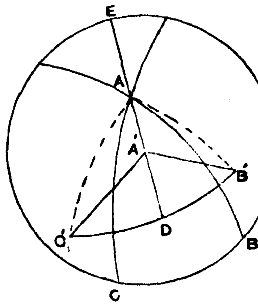
Let ABC be a spherical triangle, and A′B′C′ its polar triangle.

Then ABC will be also polar triangle to A′B′C′.

First to show that A is one of the poles of B′C′.

Join AB′, AC′ by arcs of great circles.

Then because B′ is a pole of AC, therefore B′A is a quadrant, and because C′ is a pole of AB, therefore C′A is a quadrant (§ 17).



Also B′, C′ are not at extremities of the same diameter of the sphere since B′C′ is a side of a spherical triangle.

Therefore A is one of the poles of B′C′ (§ 18).

Secondly, to show that A is that pole of B′C′ which lies on the same side of it as the opposite angle A′.

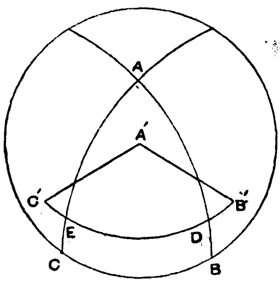
Join AA′ by an arc of a great circle and produce both ways to meet BC at E and B′C′ at D.

Since A′ is a pole of BC, therefore A′E is a quadrant,
but A and A′ lie on the same side of BC,
therefore AA′ is less than a quadrant (§ 32);
but we have just proved that A is a pole of B′C′,
hence AD is a quadrant, and AA′ being less than a quadrant it follows that A and A′ are on the same side of B′C′ (§ 32).

Similarly B is a pole of A′C′ and on the same side of it as B′, also C is a pole of A′B′ and on the same side of it as C′;
hence ABC is polar triangle to A′B′C′.

35. *The sides and angles of the polar triangle are respectively supplements of the angles and sides of the primitive triangle.*

Let a, b, c, A, B, C represent the sides and angles respectively of the primitive triangle ABC, all expressed in sexagesimal measure.



Also let a', b' c', A′, B′, C′ represent the sides and angles respectively of the polar triangle A′B′C′, all expressed in sexagesimal measure.

Now let the sides AB, AC of the primitive triangle cut the side B′C′ of the polar triangle produced if necessary at D, E.

Then because B′ is a pole of AC, B′E is a quadrant.

And because C′ is a pole of AB, C′D is a quadrant.

Therefore $B'E + C'D =$ two quadrants $= 180°$ (in sexagesimal measure),

but also $\qquad B'E + C'D = B'C' + ED$,

therefore $\qquad B'C' + ED = 180°$.

But B′C′ is denoted by a',

and ED measures A, since A is a pole of DE (§ 14).

Therefore $\qquad a' + A = 180°$.

Now this has been proved by considering A′B′C′ to be the polar triangle of ABC.

Hence since ABC is polar triangle of A′B′C′ a similar proposition must be true in this case.

Namely $\qquad a + A' = 180°$.

Similarly $\qquad b' + B = 180° = b + B'$,

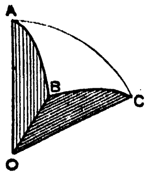
and $\qquad c' + C = 180° = c + C'$.

36. *Note.*—From these properties the polar triangle and its primitive triangle are sometimes called **supplemental triangles.** *Also it is clear that when any formula has been proved involving sides and angles of a spherical triangle, another formula, which will also be true, may be deduced by writing the supplement of the corresponding angle where a side occurs, and the supplement of the corresponding side where an angle occurs.*

37. *Any two sides of a spherical triangle are together greater than the third side.*

For any two of the three plane angles which form a solid angle are together greater than the third (Euclid xi. 20), for example, AOB + BOC > AOC.

Expressing this in circular measure, R being radius of the sphere, we get



$$\frac{AB}{R}+\frac{BC}{R}>\frac{AC}{R},$$

i.e. $AB+BC>AC.$

Similarly $AB+AC>BC,$

$BC+AC>AB.$

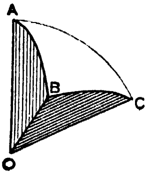
Note.—*From this proposition it is evident that any side of a spherical triangle is greater than the difference of the other two.*

$$\text{For } \left.\begin{array}{l} AB+AC>BC \\ AB+BC>AC \end{array}\right\} \therefore \left.\begin{array}{l} AB>BC-AC \\ AB>AC-BC \end{array}\right\}, \text{ i.e. } AB>BC\sim AC.$$

38. *The sum of the three sides of a spherical triangle is less than the circumference of a great circle of the sphere.*

For the sum of the three plane angles which form a solid angle is less than four right angles (Euclid xi. 21).

Therefore $AOB+BOC+AOC<$ four right angles.



Expressing this in circular measure, R being radius of the sphere, we get

$$\frac{AB}{R}+\frac{BC}{R}+\frac{AC}{R}<2\pi,$$

i.e. $AB+BC+AC<2\pi\times R,$

i.e. the sum of the three sides is less than the circumference of a great circle of the sphere.

Note.—*If the sides are expressed in sexagesimal measure, we have the sum of the three sides of a spherical triangle must be less than* 360°.

39. *The three angles of a spherical triangle are together greater than two right angles, but less than six right angles.*

Let A, B, C be the angles of a spherical triangle,

a', b', c' be the sides of the polar triangle, all expressed in circular measure.

Let R be the radius of the sphere,

then $B'C'+A'C'+A'B'<2\pi R$ (§ 38),

therefore $$\frac{B'C'}{R}+\frac{A'C'}{R}+\frac{A'B'}{R}<2\pi,$$

i.e. $$a'+b'+c'<2\pi \qquad (1),$$

but we have proved that

$$a'+A+b'+B+c'+C=3\pi \text{ (§ 35)} \qquad (2),$$

therefore $A+B+C>\pi,$

but since A, B, C are each $<\pi,$

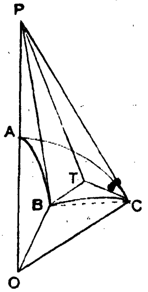
therefore $A+B+C<3\pi.$

Note.—*Expressing these results in sexagesimal measure the sum of the three angles of a spherical triangle is greater than* 180° *but less than* 540°.

40. *The angles at the base of an isosceles spherical triangle are equal to one another.*

Let ABC be a spherical triangle having the side AB equal to the side AC.

Then shall the angle ABC be equal to the angle ACB,



First, when AB, AC are less than quadrants.

Take O, the centre of the sphere, and join OA, OB, OC.

Draw BP a tangent to AB, then OBP is a right angle,

but BOA is less than a right angle, since BA is less than a quadrant.

Therefore BP produced will meet OA produced at P.

Join PC.

Then because the arc AB equals the arc AC and O is their common centre,

therefore the angle AOB equals the angle AOC (Euclid iii. 27).

Also OB equals OC for they are radii of the sphere, and OP is common.

Therefore PC equals PB and PCO = PBO (Euclid i. 4).

But PBO is a right angle (Euclid iii. 18).

Therefore PCO is a right angle and PC is a tangent to the arc AC (Euclid iii. 16).

Join BC and draw the tangents BT, CT to the arc BC.

These will meet at the point T, and BT = CT.

For OBT equals OCT, each being a right angle (Euclid iii. 18), and OBC equals OCB, since OB equals OC (Euclid i. 5).

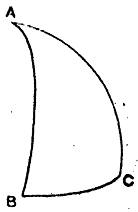
Therefore BCT and CBT are equal and less than right angles.

Hence BT, CT meet and are equal.

Join PT.

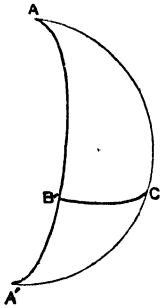
Then because PB equals PC and BT equals CT and PT is common, therefore the angle PCT equals the angle PBT (Euclid i. 8), *i.e.* the spherical angles ABC and ACB are equal.

Secondly. When AB, AC are quadrants this construction fails, but then A is pole of BC.



And therefore ABC, ACB are right angles and consequently equal (§ 17).

Thirdly. When AB, AC are greater than quadrants, the construction also fails.



In this case produce AB, AC to meet again at A′.

Then ABA′, ACA′ are semicircles (§ 10),
and therefore A′B, A′C are equal and less than quadrants.

Hence by the first case A′BC, A′CB are equal.

And consequently their supplements ABC, ACB are equal.

41. *If a spherical triangle has two equal angles, the sides opposite those angles will be equal.*

Let ABC be a spherical triangle, A′B′C′ its polar triangle, using the ordinary notation to denote sides and angles.

Let $$B = C.$$

Then we have, by § 35,
$$B + b' = 180° = C + c'.$$

Therefore $$b' = c'.$$

Consequently, by § 40,
$$B' = C',$$

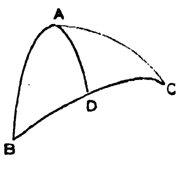
but $$B' + b = 180° = C' + c \text{ (by § 35).}$$

Therefore $$b = c.$$

42. *If one angle of a spherical triangle be greater than another, the side opposite the greater angle is greater than the side opposite the less angle.*

Let ABC be a spherical triangle having the angle BAC greater than the angle ABD,
then the side BC shall be greater than the side AC.

At the point A in the arc AB, make the spherical angle BAD equal to ABD.



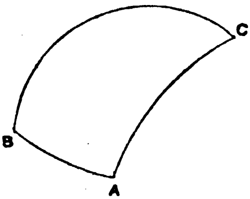
Then BD equals AD (§ 41),
therefore BD + DC = AD + DC.

But AD + DC is greater than AC (§ 37).

Therefore also BD + DC (*i.e.* BC) is greater than AC.

43. *If one side of a spherical triangle be greater than another, the angle opposite the greater side is greater than the angle opposite the less side.*

Let ABC be a spherical triangle having the side BC greater than the side AC.



Then the angle BAC shall be greater than the angle ABC.

For if BAC be not greater than ABC, BAC must be either equal to or less than ABC.

If BAC were equal to ABC, BC would be equal to AC, which is not the case.

Neither is BAC less than ABC, for then BC would be less than AC (§ 42), which is not so.

Therefore BAC is greater than ABC.

Note.—This proposition may also be established by the aid of the polar triangle.

Using the ordinary notation to denote sides and angles, suppose $a>b$ to show that A is greater than B.

We have $a>b$,

also $a+A'=180^\circ=b+B'$ (§ 35),

therefore $A'<B'$.

Consequently $a'<b'$ (§ 42),

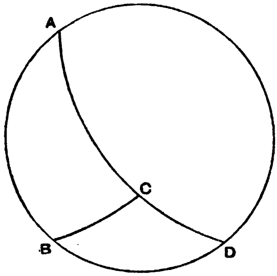
but $a'+A=180^\circ=b'+B$ (§ 35).

Consequently $A>B$.

44. *If one side AB of a spherical triangle ABC be produced, the exterior angle CBD is equal to, less or greater than the interior opposite angle BAC adjacent to that side according as the sum of the other two sides is equal to, greater or less than a semicircumference.*

Produce AC, AB to meet again at D.

Then ACD = ABD = semicircumference (§ 10).



Now if AC + CB = semicircumference AC + CD,

then CB = CD,

and ∴ CBD = CDB = BAC (§ 40).

If $AC + CB >$ semicircumference $AC + CD$,
then $CB > CD$,
and $\therefore$ $CBD < CDB$, *i.e.* $< BAC$ (§ 43).
If $AC + CB <$ semicircumference $AC + CD$,
then $CB < CD$,
and $\therefore$ $CBD > CDB$, *i.e.* $> BAC$ (§ 43).

45. The following proposition is especially useful in considering the ambiguous case of the rule of sines. It is also interesting as being the analogue of Euclid iii. 7.

Prop.—*If a point be taken in a secondary to a great circle which is not its pole,*

(1) *Of all the arcs of great circles which can be drawn from this point to the circumference of the circle the greatest is that which passes through the pole,*

(2) *and the remaining part of the semicircumference is the least arc.*

(3) *Of others that which is nearer to the arc passing through the pole is always greater than one more remote.*

(4) *And from this point to the circumference of the circle there can always be drawn two arcs which are equal to one another, and only two, one on each side of the secondary.*

(5) *From this point there can be in general drawn to the circumference on the same side of the secondary, two arcs which will cut the circumference at equal angles, and these arcs will together equal a semicircumference.*

(6) *Should the arc thus drawn be a quadrant the other arc becomes coincident with it, and the quadrant thus drawn makes the least angle any arc can make with the circumference.*

Let P be pole of the great circle BCE and A a point in the secondary EPB, and let AC be an arc nearer the pole than AD (Fig. 1).

Join PC, PD the latter cutting AC at F.

(1) Then $PC = PB$, each being a quadrant,
PA is common,
$\therefore$ $PA + PC = PA + PB$, *i.e.* $= AB$,
but $PA + PC$ two sides of APC are greater than AC,
$\therefore$ AB is greater than AC any other arc,
and hence AB is greatest of all.

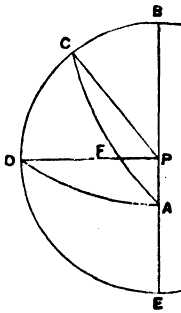


Fig. 1.

(2) $PA + AD$ two sides of $PAD > PD$ (Fig. 1),
but $PE = PD$, each being a quadrant.
$\therefore$ $PA + AD > PE$, *i.e.* $> PA + AE$.

Take away the common part PA,
then $AD > AE$.
Hence $AE <$ than any other arc AD and is therefore *least* of all.

(3) $$AF + FD > AD \text{ (Fig. 1).}$$
$$FC + FP > PC,$$
$$AF + FC + FP + FD > AD + PC,$$
$$AC + PD > AD + PC.$$
But $PD = PC$, each being a quadrant,
$$AC > AD.$$

(4) Take $EG = ED$, join AG (Fig. 2),

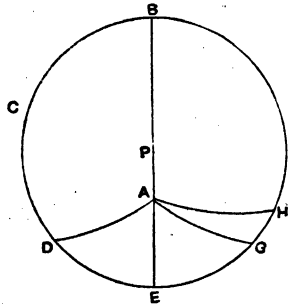


Fig. 2.

then $ED = EG$, EA is common,
and $DEA = DEG$, each being a right angle,
$\therefore AG = AD$.

And besides AG no arc can be drawn to the circumference equal to AD.

For if possible let
$$AH = AD$$
then AH also $= AG$,
i.e. an arc nearer to the one through the pole equal to one more remote, which is impossible.

(5) Draw any arc AG (Fig. 3).
Take $ED = EG$, join AD.
Produce DA to meet the circumference at H.

Then, as in (4),
$$AD = AG,$$
$$\therefore \quad AGE = ADE,$$
and since DAH is a semicircumference,
$\therefore GA + AH = DA + AH =$ semicircumference,
and $AHG = ADE = AGE$ (§ 44).

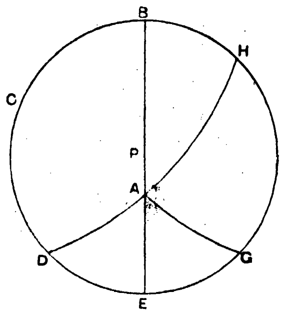


Fig. 3.

(6) Draw AD a quadrant (Fig. 4).

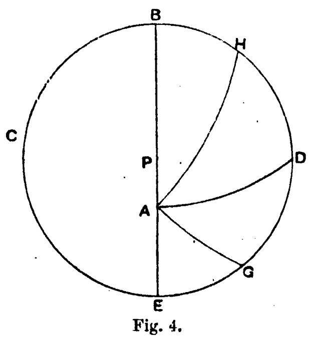


Fig. 4.

Draw also AH nearer the arc through the pole and AG more

then $\quad AG + AD < \text{semicircumference},$
therefore $\quad AGE > ADE$ (§ 44).
Also $\quad AD + AH > \text{semicircumference},$
therefore $\quad ADE < AHE$ (§ 44).

Hence ADE is the least angle which any arc drawn from A can make with the circumference.

46. *Consideration of the construction of a spherical triangle, when of two sides and the angles opposite them any three are given.*

CASE I. Given a, b, A.

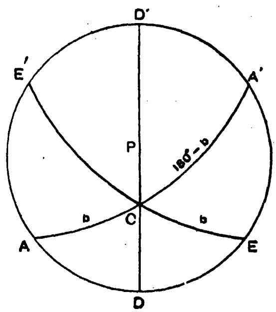
First, suppose A to be acute and b less than a quadrant.

Project the figure on the plane of the side c.

Through P the pole draw the secondary DCPD′ meeting the side c at D.

Produce AC, AD to meet at A′.

On AD produced take DE = DA.



Join EC, and produce EC to meet EA produced at E′.

Then $\quad CE = CA = b,$

$ACA' = \text{semicircle} = 180°,$

$\therefore \quad CA' = 180° - b.$

If a be greater than CA′, there can be *no* triangle (§ 27).

If a lie between CE and CA′, there can be only *one* triangle (§ 45), and B will be of *like* affection with b.

If a lie between CE and CD, there will be a corresponding value of a between AC and CD and there will therefore be *two* triangles, and the *two* values of B will be *supplemental.*

If a be less than CD, there can be *no* triangle.

Hence it is clear that if a triangle be possible there will be only *one* triangle if a lie between b and $180° - b$,
but *two* triangles if a does not lie between b and $180° - b$.

Note.—When A is obtuse and b less than a quadrant use the part of the figure CAD′A′.

When A is acute and b greater than a quadrant interchange the letters A and A′, then $CA = b$, $CA' = 180° - b$, $CE = 180° - b$.

When A is obtuse and b greater than a quadrant use the upper part of the figure after interchange of the letters A and A′.

These cases may be left for the consideration of the student.

CASE II. Given A, B, b.

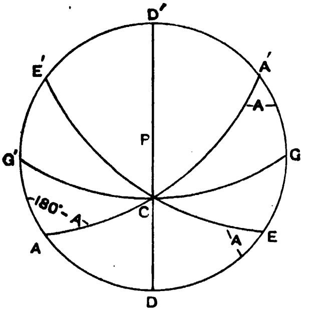
First, suppose A to be acute and b to be less than a quadrant.

Project the figure on the plane of the side c.

Through P the pole draw the secondary DCPD′ meeting the side c at D.

Produce AC, AD to meet at A′

On AD produced take DE = DA.



Join EC, and produce it to meet EA produced at E′.

Then CA′D = CED = CAD = A,

and CAD′ = 180° − A.

Draw the quadrant CG meeting AD produced at D.

If B be greater than CAD′, there can be *no* triangle.

If B lie between CED and CAD′, there can be only *one* triangle, and a will be of *like* affection with A.

If B lie between CA′D and CGD, there will be a corresponding value of B between CGD and CED, and there will therefore be *two* triangles, and the *two* values of a will be *supplemental* (§ 45 (5)).

If B be less than CGD, there can be *no* triangle.

Hence it is clear that if a triangle be possible there will be only *one* triangle, if B lie between A and 180° − A, but *two* triangles, if B does not lie between A and 180° − A.

Note.—When A is obtuse and b less than a quadrant, use the part of the figure CAD′A′.

When A is acute and b greater than a quadrant, interchange the letters A and A′; then

CAD′ = 180° − A, CED′ = 180° − A, CAD = A.

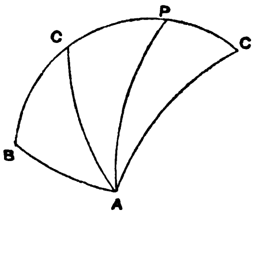
When A is obtuse and b greater than a quadrant, use the upper part of the figure after interchange of the letters A and A′.

These cases may be left for the consideration of the student.

47. *In a right-angled spherical triangle the angles adjacent to the hypotenuse are of like affection with the opposite sides.*

Let ABC be a right-angled triangle right angled at B.

On BC produced if necessary take BP a quadrant and join PA.

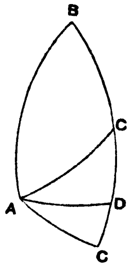


Then P is pole of AB (§ 19), and therefore PAB is a right angle.

Also it is clear that CAB is greater or less than a right angle according as CB is greater or less than a quadrant.

Similarly for the angle ACB and the side AB

48. *In a quadrantal triangle the angles adjacent to the quadrant are of like affection with the opposite sides.*



Let ABC be a quadrantal triangle, AB being the quadrant.

At the point A in the arc BA make BAD a right angle.

Then B is pole of AD (§ 19), and therefore BD is a quadrant.

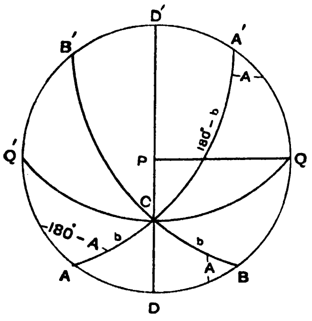
And it is clear that BC is greater or less than the quadrant BD, according as BAC is greater or less than the right angle BAD.

Similarly for the angle ABC and the side AC.

49. The accompanying figure will enable the student to practically read off many of the properties of spherical triangles which have been formally proved.

P is pole of the primary circle AB.

Hence DPD′ will be a secondary, cutting the primary circle at right angles.



ACA′, BCB′ are great circles cutting the primary circle obliquely.

If DQ be taken, a quadrant Q will be pole of DCD′ and QC a quadrant.

Hence if QC be produced, Q′ is pole of DCD′ and CQ′ will be a quadrant.

(1) The greater side is opposite the greater angle and conversely.

In the triangle A′CB, A′C is greater than CB, and the angle A′BC is greater than the angle BA′C.

(2) A spherical triangle may have three acute angles, *e.g.* the triangle ABC.

A spherical triangle may have three right angles, *e.g.* the triangle PQD.

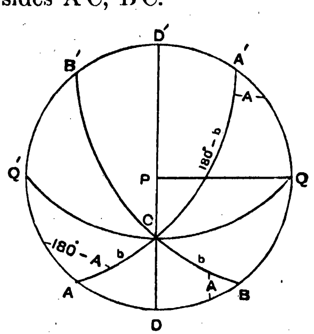
A spherical triangle may have three obtuse angles, *e.g.* the triangle A′CQ′.

(3) When two angles of a spherical triangle are of *like* affection, the perpendicular from the third angle to the opposite side falls *inside* the triangle and is of *like* affection with those angles, *e.g.* In the triangle ABC, A and B are *acute*, and the perpendicular CD, which is *less* than a quadrant, falls *inside* the triangle—In the triangle A′B′C, A′ and B′ are *obtuse*, and the perpendicular CD′, which is greater than a quadrant, falls *inside* the triangle.

(4) When two angles of a spherical triangle are of *unlike*

affection, the perpendicular from the third angle to the opposite side falls *outside* the triangle opposite the acute angle, and is of *like* affection with that angle. Thus in the triangle A′CB the perpendicular CD is of *like* affection with A′, and falls outside the triangle opposite the acute angle A′.

(5) When two sides of a spherical triangle are of *like* affection, the quadrant from the third angle to the opposite side falls outside the triangle, and the angle which it makes with the third side (produced) is of like affection with the two sides of the triangle. Thus in the triangle ABC, CQ falls outside the triangle, and CQA is of like affection with AC and CB. In the triangle A′B′C, CQ falls outside the triangle, and CQB′ is of like affection with the two sides A′C, B′C.



(6) When two sides of a spherical triangle are of unlike affection, the quadrant drawn from the third angle to the opposite side falls inside the triangle, and the angles which it makes with that side are of like affection with the sides of the triangle opposite to them. Thus in the triangle A′CB, the quadrant falls inside the triangle, the sides A′C, CB, being of unlike affection; also the angle A′QC is greater than a right angle, whilst the side A′C is greater than a quadrant. Similarly BQC is less than a right angle, and BC is less than a quadrant.

(7) If we have to construct a spherical triangle having given two sides and an angle opposite one of these sides, for example, given a, b, A, we must remember that if a lie between b and $180° - b$, there can be only one triangle and B will be of like affection with b; but if a does not lie between b and $180° - b$, there will be two triangles having the given parts, and the two values of B will be supplemental.

(8) If the parts given be two angles and a side opposite one of them, for example, A, B, b, then if B lie between A and $180° - A$, there will be only one triangle and a will be of like affection with A; but if B does not lie between A and $180° - A$, there will be two triangles having the given parts, and the two values of a will be supplemental.

(9) If in a right-angled triangle a side and the opposite angle are the only other parts given, the triangle is ambiguous, *i.e.* there are two triangles having the given parts. Suppose CAD and CD be given to solve the right-angled triangle ACD. Reference to the figure will show that the triangle A′CD has these same parts, and

its other parts will be supplements of corresponding parts of ACD. Thus $A'D = 180° - AD$, $A'C = 180° - AC$, $A'CD = 180° - ACD$.

(10) If in a quadrantal triangle a side and the opposite angle are the only other parts given, the triangle is ambiguous. In the quadrantal triangle ACQ, CQ being the quadrant, suppose the other parts given to be CQA and CA. The triangle ACQ′ has the same parts, and its other parts will be supplements of the corresponding parts of ACQ. Thus $AQ' = 180° - AQ$, $CAQ' = 180° - CAQ$, $ACQ' = 180° - ACQ$.

(11) In a right-angled and also in a quadrantal triangle, a side and the angle opposite it are of *like* affection. In the right-angled triangle A′CD, CA′D is acute and CD is less than a quadrant, A′CD is obtuse and A′D is greater than a quadrant, and so on. In the quadrantal triangle ACQ, CQA is acute and AC is less than a quadrant, ACQ is obtuse and AQ greater than a quadrant.

(12) In an isosceles triangle the perpendicular bisects the vertical angle and the opposite side. Thus CD bisects the angle ACB and the base AB; CD′ bisects the angle A′CB′ and the base A′B′.

(13) When two sides of a spherical triangle are supplemental the angles opposite these aides are supplemental, and the quadrant drawn from the third angle to the opposite side bisects the third angle and third side. Thus in the triangle A′CB,

$$CB\ (=CA) + CA' = 180° \text{ and } CBA'\ (=180° - A) + CA'B = 180°.$$

The quadrant CQ bisects A′CB, for $A'CQ = ACQ' = BCQ$, also CQ bisects A′B, for the arc $Q'A'Q = 180° =$ the arc $A'Q'A$.

Take away from each the arc A′Q′ and we have

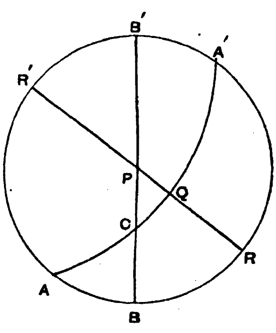
$$AQ' = A'Q,$$

but $$AQ' = BQ,$$

therefore $$A'Q = BQ.$$

(14) *In a right-angled triangle if A be acute* a *is less than A, but if A be obtuse* a *is greater than A.*

From P the pole of AB draw PQ perpendicular to AC and produce it to meet AB at R,



then A is pole of PR (§ 19), and therefore $QR = A$ (§ 14).

Also PQ is the least arc from P to AC and is therefore less than PC (§ 45 (2)).

Hence CB is always less than QR, *i.e.* a is always less than A, if A be acute.

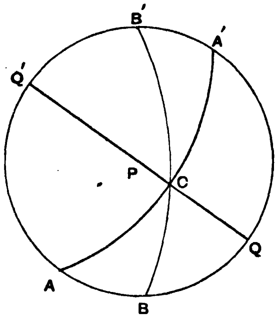
Similarly in the triangle A′CB′, where A′ is obtuse, CB′ or a' may be shown to be greater than A′.

(15) *In a quadrantal triangle if A be acute* a *is greater than A, but if A be obtuse* a *is less than A.*

Let AC be the quadrant.

From P the pole of AB, draw PCQ.

Then because AP, AC are quadrants, A is pole of PCQ (§ 18), and therefore CQ = A (§ 14).



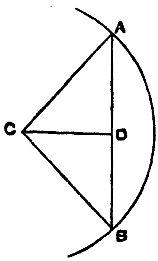
But CQ is the least arc from C to BA (§ 45 (2)); and therefore CQ is always less than CB, *i.e.* a is always greater than A, if A be acute.

Similarly

CQ′ is always greater than CB′

or a' is always less than A′ if A′ be obtuse.

50. It is sometimes necessary to find the length of the chord of an arc, having given the number of degrees in the arc and the radius of the circle.



From C the centre of the circle of which AB is an arc,

draw CD perpendicular to the chord AB,

then $\dfrac{\text{chord AB}}{\text{AC}} = 2\dfrac{\text{AD}}{\text{AC}} = 2 \sin \dfrac{\text{ACB}}{2}$.

So that if R be the radius we have

$$\text{chord} = 2\text{R} \sin \frac{\text{arc}}{2}.$$

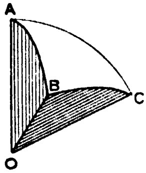
Example. Find the chord of $20°$ in a circle of radius 6 inches.

Here chord $= 12 \sin 10°$.

log. 12	1·079181
log. sin 10°	$\bar{1}$·239670
log. chord	0·318851
chord =	2·084 inches.

51. If the length of an arc of a circle be given in linear units, the number of degrees in the arc can be found, provided the length of the radius of the circle is known. Conversely, we can find the length of an arc of a circle, if we know the value of the arc in degrees and the length of the radius of the circle.

Ex. 1. Suppose the arc AB is 3 inches long and the radius of the circle is 5 inches.



$$\left.\begin{aligned}&\text{Then circular measure } AOB = \frac{\text{arc } AB}{OA} = \frac{3}{5} \quad (1)\\ &\text{also} \quad \text{circular measure } AOB = \frac{AOB}{57^\circ{\cdot}29577} \quad (2)\end{aligned}\right\}$$

therefore $AOB = \frac{3}{5}$ of $57^\circ{\cdot}29577 = \frac{171^\circ{\cdot}88731}{5} = 34^\circ{\cdot}37746.$

Hence $\quad$ arc $AB = 34^\circ{\cdot}37746.$

Ex. 2. Suppose the arc $AB = 20^\circ\ 15'$ and the radius of the circle is 8 inches.

$$\left.\begin{aligned}&\text{Then circular measure } AOB = \frac{\text{arc } AB}{OA} = \frac{\text{arc } AB}{8} \quad (1)\\ &\text{also} \quad \text{circular measure } AOB = \frac{20\frac{1}{4}}{57{\cdot}29577} \quad (2)\end{aligned}\right\}$$

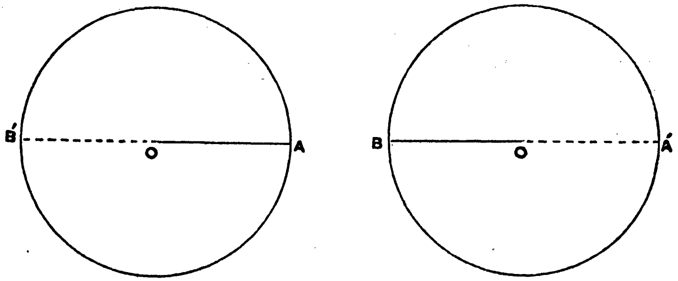
therefore $\quad \frac{\text{arc } AB}{8} = \frac{20\frac{1}{4}}{57{\cdot}29577},$

or $\quad$ arc $AB = \frac{162}{57{\cdot}29577} = 2{\cdot}827$ inches.

52. Models are of great use in helping the student to understand the principles of spherical geometry. Out of a sheet of paper or thin cardboard, the more simple models may be readily made. The following hints may be useful :

(1) Two great circles bisect each other.

(2) Only one great circle can join two points on a sphere, unless the points are at extremities of the same diameter.



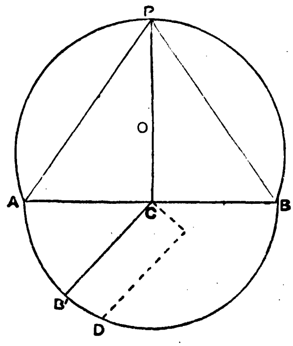
Describe two circles.

Cut round their circumferences.

Also cut along the radii AO, BO.

Slip the cut BO along AO, till B, B′, A, A′ coincide, and we have a model of the figures required.

(3) A pole of a circle is equally distant from every point in the circumference of the circle.



Describe with O as centre any arc of a circle APB.

Join AB, and bisect AB in C.

Join CO and produce to meet the circumference at P; join PA, PB. With C as centre and radius CA describe an arc AB′B.

Make ACB′ any angle.

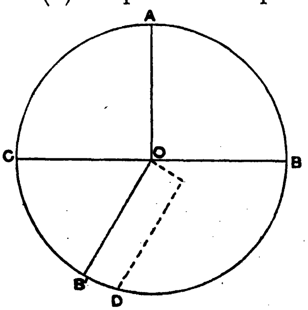
Cut round the arcs APB, AB′B.

Cut along BC and along the dotted line.

Fold the paper along PC, AC, B′C.

Bring the free edge CB to CB′, and gum down the flap CD.

(4) Propositions respecting the pole of a great circle.



Describe a circle.

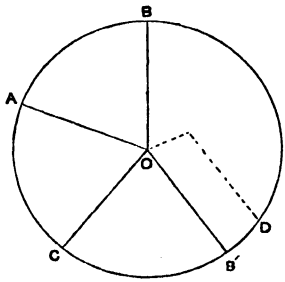
At the centre O make BOA, AOC right angles, and COB′ any angle.

Cut round the circumference, also along BO and along the dotted line.

Fold the paper along AO, CO, B′O.

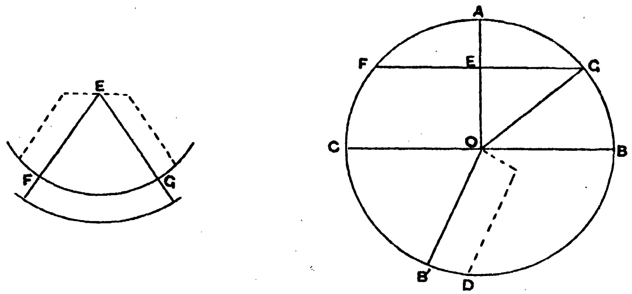
Bring the free edge BO to B′O, and gum down the flap OD.

(5) A spherical triangle and the solid angle with which it is connected.



The same construction as (4), except that the three angles may be of any magnitude.

(6) To compare the arc of a small circle with the arc of a great circle subtending the same angle at their respective centres.



This is the same as (4) with the addition.

Draw a chord FG at right angles to OA. Join OG.

With radius EF describe an arc FG.

Make the angle FEG = COB′.

Cut along the dotted line and along the arc FG.

Fold the paper along EF, FG, and slip the angle FEG into the model (4), so that EF, EG of the angle coincide with EF, EG of the model. Gum down the flaps.

CHAPTER IV

RELATIONS BETWEEN THE TRIGONOMETRICAL RATIOS OF THE SIDES AND ANGLES OF SPHERICAL TRIANGLES

53. Every spherical triangle has six parts, viz. three angles and three sides.

An equation between any four of these parts must connect the sides and angles.

The number of such equations we can form will be the same as the number of combinations of six things taken four at a time, and therefore

$$= \frac{\lfloor 6}{\lfloor 4 \lfloor 2} = \frac{30}{2} = 15.$$

54. But in selecting four out of the six parts of a triangle the two parts omitted may be

(1) two angles,
(2) two sides,
(3) a side and the angle opposite it,
(4) a side and one of the angles adjacent to it.

(1) If two angles be omitted the equations will evidently be similar whichever angle be retained, so that the three particular equations in this case reduce to one general form.

(2) If two sides be omitted the equations will evidently be similar whichever side be retained, so that the three particular equations in this case will reduce to one general form.

(3) If a side and the angle opposite it be omitted the equations will evidently be similar whichever side and angle opposite it be omitted, and the three particular equations will in this case also reduce to one general form.

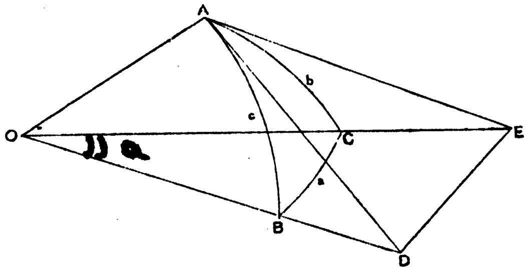
(4) If a side and an angle adjacent to it be omitted, since each of the three sides has two angles adjacent to it, this omission may be made in six ways, and the resulting six equations will all be similar and thus reduce to one general form.

55. Hence the *fifteen particular equations* we get for the solution of spherical triangles reduce to *four general equations*, viz.

(1) *An equation between an angle and the three sides.*
(2) *An equation between a side and the three angles.*
(3) *An equation between two sides and the opposite angles.*
(4) *An equation between two sides and two angles all lying together.*

56. (1) *To find the equation between an angle and the three sides i.e. between* (A, a, b, c); (B, a, b, c); (C, a, b, c).

Let ABC be a spherical triangle.



(1) When AB, AC are less than quadrants.

From O the centre of the sphere draw OA, OB, OC.

Draw AD, AE tangents to the arcs AB, AC respectively—then DAE = A.

Now OAD, OAE are right angles,
but AOB, AOC are less than right angles, since AB, AC are less than quadrants,
therefore AD, OB, if produced, will meet at D.
Similarly AE, OC, if produced, will meet at E.

Join DE.

$$DE^2 = OE^2 + OD^2 - 2OE \cdot OD \cos a \qquad (1),$$
$$DE^2 = AE^2 + AD^2 - 2AE \cdot AD \cos A \qquad (2).$$

Subtracting (2) from (1)

$$0 = (OE^2 - AE^2) + (OD^2 - AD^2) - 2OE \cdot OD \cos a + 2AE \cdot AD \cos A$$

or $$0 = 2OA^2 - 2OE \cdot OD \cos a + 2AE \cdot AD \cos A.$$

Dividing by $2OE \cdot OD$ we get

$$0 = \frac{OA \cdot OA}{OE \cdot OD} - \cos a + \frac{AE \cdot AD}{OE \cdot OD} \cos A$$

or $$0 = \cos b \cos c - \cos a + \sin b \sin c \cos A$$

i.e. $$\cos a = \cos b \cos c + \sin b \sin c \cos A \qquad (1)$$

or $$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} \qquad (2)$$

Similarly

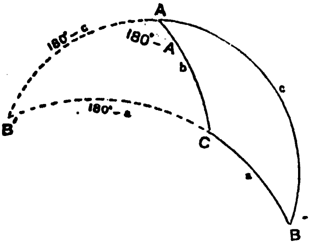
$$\cos b = \cos a \cos c + \sin a \sin c \cos B \qquad (1)$$

or $$\cos B = \frac{\cos b - \cos a \cos c}{\sin a \sin c} \qquad (2)$$

and

$$\left.\begin{aligned}\cos c &= \cos a \cos b + \sin a \sin b \cos C \qquad (1)\\ \text{or} \quad \cos C &= \frac{\cos c - \cos a \cos b}{\sin a \sin b} \qquad (2)\end{aligned}\right\}$$

(2) When one of the containing sides, *e.g.* AB, is greater than a quadrant.



Produce BA, BC to meet again at B′, then BAB′, BCB′ are semicircles, and ∴ AB′, AC are less than quadrants.

In the triangle B′AC,
B′AC = 180° − A, AB′ = 180° − c, B′C = 180° − a.

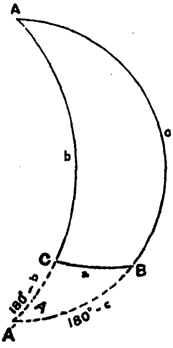
$$\cos B'AC = \frac{\cos B'C - \cos B'A \cos AC}{\sin B'A \sin AC}$$

or $$\cos(180° - A) = \frac{\cos(180° - a) - \cos(180° - c)\cos b}{\sin(180° - c)\sin b},$$

i.e. $$-\cos A = \frac{-\cos a + \cos b \cos c}{\sin b \sin c}$$

or $$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

(3) When both of the containing sides AB, AC are greater than quadrants.



Produce AB, AC to meet again at A′.
Then ABA′, ACA′ are semicircles,
∴ A′B, A′C are less than quadrants.
In the triangle A′BC,
A′C = 180° − b, A′B = 180° − c, BA′C = A.

$$\cos BA'C = \frac{\cos BC - \cos A'C \cos A'B}{\sin A'C \sin A'B}$$

or $$\cos A = \frac{\cos a - \cos(180° - b)\cos(180° - c)}{\sin(180° - b)\sin(180° - c)}$$

$$= \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

(4) When one of the containing sides *e.g.* AC, is a quadrant, suppose AB to be greater than a quadrant (Fig. 1), suppose AB to be less than a quadrant (Fig. 2).

On AB, produced if necessary, take AD a quadrant. Join CD.
Then A is pole of CD, and therefore CD = A.
Also CDB is a right angle.

If CD be a quadrant.

C is pole of AB, and we have $a=90^\circ$, $A=90^\circ$, $b=90^\circ$.

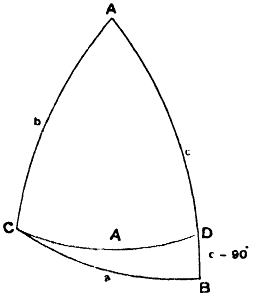


Fig. 1.

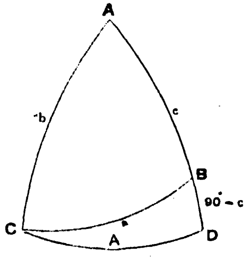


Fig. 2.

Now if we substitute these values in the formula

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

we get

$$\cos 90^\circ = \frac{\cos 90^\circ - \cos 90^\circ \cos c}{\sin 90^\circ \sin c}$$

or $$0=0.$$

Hence the fundamental formula is true for the triangle ABC in such a case.

But if CD be not a quadrant.

In the triangle BDC we have

$$\cos BC = \cos CD \cos DB + \sin CD \sin DB \cos CDB,$$

i.e. $$\cos a = \cos A \cos (90^\circ \sim c) + \sin A \sin (90^\circ \sim c) \cos 90^\circ$$
$$= \cos A \sin c.$$

Now if in the formula

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

we write $b=90^\circ$, we get

$$\cos a = \cos 90^\circ \cos c + \sin 90^\circ \sin c \cos A$$
$$= \sin c \cos A,$$

which shows that the formula is true for the triangle ABC.

(5) When both the containing sides AB, AC are quadrants.

Then A is pole of BC.

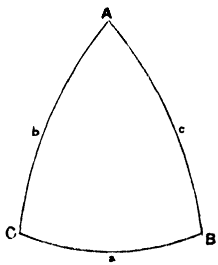
Therefore BC $(=a) = A$.

Now if in the formula

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

we write $b=90^\circ$, $c=90^\circ$, we get

$$\cos A = \frac{\cos a - \cos 90^\circ \cos 90^\circ}{\sin 90^\circ \sin 90^\circ} = \cos a.$$



Hence in all cases, whatever be the values

of the containing sides, the formula $\cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}$ is true.

This is called the fundamental formula of spherical trigonometry, because from it all the rules for the solution of spherical triangles may be deduced. It is also often referred to as the formula for the cosine of an angle of a spherical triangle in terms of functions of the sides.

57. To find the equation between a side and the three angles, *i.e.* between (a, A, B, C); (b, A, B, C); (c, A, B, C).

Let a, b, c, A, B, C be the sides and angles respectively of the spherical triangle ABC.

And let a', b', c', A', B', C' be the sides and angles respectively of its polar triangle $A'B'C'$, all supposed to be expressed in sexagesimal measure.

Then, by § 56,

$$\cos A' = \frac{\cos a' - \cos b' \cos c'}{\sin b' \sin c'},$$

or

$$\cos (180^\circ - a) = \frac{\cos (180^\circ - A) - \cos (180^\circ - B) \cos (180^\circ - C)}{\sin (180^\circ - B) \sin (180^\circ - C)} \quad (\S\ 35),$$

i.e.

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}. \ (+)$$

Similarly

$$\cos b = \frac{\cos B + \cos A \cos C}{\sin A \sin C},$$

and

$$\cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B}.$$

58. To find the equation between two sides and the angles opposite them,
i.e. between (A, a, B, b); (A, a, C, c); (B, b, C, c).

$$\frac{\sin A}{\sin a} = \frac{\sqrt{1 - \cos^2 A}}{\sin a} = \frac{\sqrt{1 - \left(\dfrac{\cos a - \cos b \cos c}{\sin b \sin c}\right)^2}}{\sin a}$$

$$= \sqrt{\frac{\sin^2 b \sin^2 c - \cos^2 a - \cos^2 b \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 a \sin^2 b \sin^2 c}}$$

$$= \sqrt{\frac{(1 - \cos^2 b)(1 - \cos^2 c) - \cos^2 a - \cos^2 b \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 a \sin^2 b \sin^2 c}}$$

$$= \frac{\sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}}{\sin a \sin b \sin c}.$$

Similarly

$\dfrac{\sin B}{\sin b}$ and $\dfrac{\sin C}{\sin c}$ may each be shown to be equal to the same expression.

Therefore

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

Note.—In solving a spherical triangle this formula is often referred to as the Rule of Sines.

59. To find the equation between two sides and two angles, all lying together,
i.e. between

(A, b, C, a), (b, C, a, B), (C, a, B, c), (a, B, c, A), (B, c, A, b), (c, A, b, C).

By § 56

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Now eliminate c by writing

$$\cos c = \cos a \cos b + \sin a \sin b \cos C \text{ (§ 56)},$$

and

$$\sin c = \frac{\sin a \sin C}{\sin A} \text{ (§ 58)},$$

and we get

$$\cos a = \cos b (\cos a \cos b + \sin a \sin b \cos C) + \frac{\sin a \sin b \sin C \cos A}{\sin A}$$

$$= \cos a \cos^2 b + \sin a \sin b \cos b \cos C + \sin a \sin b \cot A \sin C,$$

$$\therefore \cos a (1 - \cos^2 b) = \sin a \sin b \cos b \cos C + \sin a \sin b \cot A \sin C$$

or

$$\frac{\cos a \sin^2 b}{\sin a \sin b} = \cos b \cos C + \cot A \sin C,$$

i.e.

$$\cot a \sin b = \cos b \cos C + \cot A \sin C$$

or

$$\cos b \cos C = \cot a \sin b - \cot A \sin C.$$

60. This formula may be remembered by noting that the parts A, b, C, a all lie together; of these

a is the outer side, A the outer angle,
b is the inner side, C the inner angle.

Then the product of the cosines of the inner side and inner angle equals the product of cot outer side, into sine inner side less the product of cot outer angle, into sine inner angle.

The two *o*'s in *cot outer* followed by the two *i*'s in *sine inner* will aid the memory.

This rule will enable the student to write down the other five equations, connecting four parts of a triangle all lying together.

Another way of noting how the parts lie is that they are two sides and the included angle (b, C, a), and an angle opposite one of those sides (A or B).

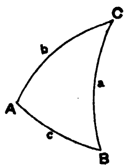
CHAPTER V

SOLUTION OF RIGHT-ANGLED SPHERICAL TRIANGLES AND QUADRANTAL TRIANGLES

61. The four general equations established in §§ 56-59 are sufficient for the complete solution of *right-angled spherical triangles* and *quadrantal triangles.* If in the general formulæ we substitute the values of the ratios of 90° which occur, the resulting equations in every case will be found to be adapted for use with logarithms.

I. Solution of Right-Angled Spherical Triangles

62. Let ABC be a right-angled spherical triangle, right-angled at A.



(1) The equation between A, a, b, c is

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c},$$

when $A = 90°$, $\cos A = 0$,

and this equation becomes

$$\cos a = \cos b \cos c \qquad (1).$$

(2) The equation between a, A, B, C is

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C},$$

when $A = 90°$, $\cos A = 0$,

and this equation becomes

$$\cos a = \cot B \cot C \qquad (2).$$

(3) The equation between b, A, B, C is

$$\cos b = \frac{\cos B + \cos A \cos C}{\sin A \sin C},$$

when $A = 90°$, $\cos A = 0$, $\sin A = 1$,

and this equation becomes

$$\cos B = \cos b \sin C \qquad (3).$$

(4) The equation between c, A, B, C is

$$\cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B},$$

when $$A = 90^\circ,\ \cos A = 0,\ \sin A = 1,$$
and this equation becomes
$$\cos C = \cos c \sin B \qquad (4).$$

(5) The equation between A, a, B, b is
$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b},$$
when $$A = 90^\circ,\ \sin A = 1,$$
and this equation becomes
$$\sin b = \sin a \sin B \qquad (5).$$

(6) The equation between A, a, C, c is
$$\frac{\sin A}{\sin a} = \frac{\sin C}{\sin c},$$
when $$A = 90^\circ,\ \sin A = 1,$$
and this equation becomes
$$\sin c = \sin a \sin C \qquad (6).$$

(7) The equation between A, b, C, a is
$$\cos b \cos C = \cot a \sin b - \cot A \sin C,$$
when $$A = 90^\circ,\ \cot A = 0,$$
and this equation becomes
$$\cos C = \cot a \tan b \qquad (7).$$

(8) The equation between c, A, b, C is
$$\cos b \cos A = \cot c \sin b - \cot C \sin A,$$
when $$A = 90^\circ,\ \cos A = 0,\ \sin A = 1,$$
and this equation becomes
$$\cot C = \cot c \sin b$$
or $$\sin b = \cot C \tan c \qquad (8).$$

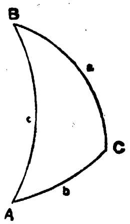
(9) The equation between B, c, A, b is
$$\cos c \cos A = \cot b \sin c - \cot B \sin A,$$
when $$A = 90^\circ,\ \cos A = 0,\ \sin A = 1,$$
and this equation becomes
$$\cot B = \cot b \sin c$$
or $$\sin c = \cot B \tan b \qquad (9).$$

(10) The equation between a, B, c, A is
$$\cos c \cos B = \cot a \sin c - \cot A \sin B,$$
when $$A = 90^\circ,\ \cot A = 0,$$
and this equation becomes
$$\cos c \cos B = \cot a \sin c$$
or $$\cos B = \cot a \tan c \qquad (10).$$

II. Solution of Quadrantal Triangles

63. Let ABC be a quadrantal triangle, BC (or a) being the quadrant.

(1) The equation between a, A, B, C is
$$\cos a = -\frac{\cos A + \cos B \cos C}{\sin B \sin C},$$



when $a=90$, $\cos a=0$,
and this equation becomes

$$\cos A = -\cos B \cos C \qquad (1).$$

(2) The equation between A, a, b, c is

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c},$$

when $a=90^\circ$, $\cos a=0$,
and this equation becomes

$$\cos A = -\cot b \cot c \qquad (2).$$

(3) The equation between B, a, b, c is

$$\cos B = \frac{\cos b - \cos a \cos c}{\sin a \sin c},$$

when $a=90^\circ$, $\cos a=0$, $\sin a=1$,
and this equation becomes

$$\cos b = \sin c \cos B \qquad (3).$$

(4) The equation between C, a, b, c is

$$\cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b},$$

when $a=90^\circ$, $\cos a=0$, $\sin a=1$,
and this equation becomes

$$\cos c = \sin b \cos C \qquad (4).$$

(5) The equation between A, a, B, b is

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b},$$

when $a=90^\circ$, $\sin a=1$,
and this equation becomes

$$\sin B = \sin b \sin A \qquad (5).$$

(6) The equation between A, a, C, c is

$$\frac{\sin A}{\sin a} = \frac{\sin C}{\sin c},$$

when $a=90^\circ$, $\sin a=1$,
and this equation becomes

$$\sin C = \sin c \sin A \qquad (6).$$

(7) The equation between A, c, B, a is

$$\cos c \cos B = \cot a \sin c - \cot A \sin B,$$

when $a=90^\circ$, $\cot a=0$,
and this equation becomes

$$\cos c \cos B = -\cot A \sin B$$

or

$$\cos c = -\cot A \tan B \qquad (7).$$

(8) The equation between c, B, a, C is

$$\cos a \cos B = \cot c \sin a - \cot C \sin B,$$

when $a=90^\circ$, $\cos a=0$, $\sin a=1$,
and this equation becomes

$$\cot c = \cot C \sin B$$

or

$$\sin B = \cot c \tan C \qquad (8).$$

(9) The equation between B, a, C, b is

$$\cos a \cos C = \cot b \sin a - \cot B \sin C,$$

when $$a = 90^\circ, \cos a = 0, \sin a = 1,$$

and this equation becomes

$$\cot b = \cot B \sin C$$

or $$\sin C = \cot b \tan B \qquad (9).$$

(10) The equation between a, C, b, A is

$$\cos b \cos C = \cot a \sin b - \cot A \sin C,$$

when $$a = 90^\circ, \cot a = 0,$$

and this equation becomes

$$\cos b \cos C = -\cot A \sin C$$

or $$\cos b = -\cot A \tan C \qquad (10).$$

Statement of Napier's Rules for the Solution of Right-angled Spherical and Quadrantal Triangles

64. Napier has embodied the ten equations for the solution of right-angled triangles, and the ten equations for the solution of quadrantal triangles, in the following rules :

I. Right-angled Triangles

Leaving the right angle out of consideration, there remain five parts of the triangle, viz. the three sides and the two remaining angles.

Circular parts.—The two sides which contain the right angle, the complement of the side opposite the right angle and the complements of the two remaining angles are called the *circular parts.*

If we imagine these five circular parts to be arranged round a circle in the same order as they come in the triangle, we shall find that if *any three* circular parts be taken one of them may be so chosen that the other two are either both *adjacent* to it or else both *opposite* to it.

The part so selected is called the **middle part,** and the other two are called either **adjacent parts** or **opposite parts.**

These data being understood Napier's rules are

(1) sine *middle* part equals product of the *tangents* of *adjacent* parts,

(2) sine *middle* part equals product of the *cosines* of *opposite* parts.

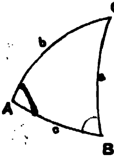
II. Quadrantal Triangles

Circular parts.—The quadrant being left out of consideration, the complement of the angle opposite the quadrant, the two remaining angles, and the complements of the two remaining sides are the *circular parts* in this case.

The same rules apply as in right-angled triangles with the addition of the following rule:

(3) When the *adjacent* parts or the *opposite* parts are *both* sides or *both* angles, the product of their ratios must be considered *negative*.

Application of Napier's Rules to the Solution of Right-angled Spherical Triangles



65. Let ABC be a right-angled triangle right-angled at A.

The circular parts are

$$90° - a,\ 90° - B,\ 90° - C,\ b,\ c.$$ no order.

Taking each of these circular parts in succession as middle part, we have

When $(90° - a)$ is *middle* part, $(90° - B)$, $(90° - C)$ are *adjacent* parts, b, c are *opposite* parts,

then

$$\left.\begin{aligned} \sin(90° - a) &= \tan(90° - B)\tan(90° - C) \text{ or } \cos a = \cot B \cot C \quad &(1)\\ \sin(90° - a) &= \cos b \cos c \qquad \cos a = \cos b \cos c. \quad &(2)\end{aligned}\right\}$$

When $(90° - B)$ is *middle* part, $(90° - a)$, c are *adjacent* parts, b, $(90° - C)$ are *opposite* parts,

then

$$\left.\begin{aligned} \sin(90° - B) &= \tan(90° - a)\tan c \text{ or } \cos B = \cot a \tan c \quad &(3)\\ \sin(90° - B) &= \cos b \cos(90° - C) \qquad \cos B = \cos b \sin C. \quad &(4)\end{aligned}\right\}$$

When c is *middle* part, $(90° - B)$, b are *adjacent* parts, $(90° - a)$, $(90° - C)$ are *opposite* parts,

then

$$\left.\begin{aligned} \sin c &= \tan(90° - B)\tan b \text{ or } \sin c = \cot B \tan b \quad &(5)\\ \sin c &= \cos(90° - a)\cos(90° - C) \qquad \sin c = \sin a \sin C. \quad &(6)\end{aligned}\right\}$$

When b is *middle* part, $(90° - C)$, c are *adjacent* parts, $(90° - a)$, $(90° - B)$ are *opposite* parts,

then

$$\left.\begin{aligned} \sin b &= \tan(90° - C)\tan c \text{ or } \sin b = \cot C \tan c \quad &(7)\\ \sin b &= \cos(90° - a)\cos(90° - B) \qquad \sin b = \sin a \sin B. \quad &(8)\end{aligned}\right\}$$

When $(90° - C)$ is *middle* part, $(90° - a)$, b are *adjacent* parts, c, $(90° - B)$ are *opposite* parts,

then

$$\left.\begin{aligned} \sin(90° - C) &= \tan(90° - a)\tan b \text{ or } \cos C = \cot a \tan b \quad &(9)\\ \sin(90° - C) &= \cos c \cos(90° - B) \qquad \cos C = \cos c \sin B. \quad &(10)\end{aligned}\right\}$$

Verification of the Ten Equations obtained by the Application of Napier's Rules to Right-angled Spherical Triangles

66. These ten equations have been proved to be true in § 62. They may be verified as follows:

Write down the equation connecting the three parts in any formula with the right angle, and substitute the values of the ratios of 90° which occur,

e.g. when A is a right angle, prove $\sin c = \cot B \tan b$.

The equation between B, c, A, b is

$$\cos c \cos A = \cot b \sin c - \cot B \sin A,$$

when $A = 90^\circ$, $\cos A = 0$, $\sin A = 1$,

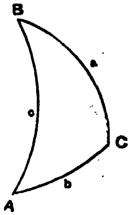
and the equation reduces to

$$\cot b \sin c = \cot B$$

or $$\sin c = \cot B \tan b.$$

Similarly any of the other equations of § 65 may be verified.

Application of Napier's Rules to the Solution of Quadrantal Triangles



67. Let ABC be a quadrantal triangle, BC (or a) being the quadrant.

The circular parts are

$$90^\circ - A,\ 90^\circ - b,\ 90^\circ - c,\ B,\ C.$$

Taking each of these circular parts in succession as middle part, we have

When $(90^\circ - A)$ is *middle* part, $(90^\circ - b)$, $(90^\circ - c)$ are *adjacent* parts, B, C are *opposite* parts,

then

$$\left.\begin{array}{ll} \sin(90^\circ - A) = -\tan(90^\circ - b)\tan(90^\circ - c) & \text{or } \cos A = -\cot b \cot c \quad (1) \\ \sin(90^\circ - A) = -\cos B \cos C & \cos A = -\cos B \cos C. \quad (2) \end{array}\right\}$$

When $(90^\circ - b)$ is *middle* part, C, $(90^\circ - A)$ are *adjacent* parts, $(90^\circ - c)$, B are *opposite* parts,

then

$$\left.\begin{array}{ll} \sin(90^\circ - b) = -\tan C \tan(90^\circ - A) & \text{or } \cos b = -\tan C \cot A \quad (3) \\ \sin(90^\circ - b) = \cos(90^\circ - c)\cos B & \cos b = \sin c \cos B. \quad (4) \end{array}\right\}$$

When C is *middle* part, $(90^\circ - b)$, B are *adjacent* parts, $(90^\circ - A)$, $(90^\circ - c)$ are *opposite* parts,

then

$$\left.\begin{array}{ll} \sin C = \tan(90^\circ - b)\tan B & \text{or } \sin C = \cot b \tan B \quad (5) \\ \sin C = \cos(90^\circ - A)\cos(90^\circ - c) & \sin C = \sin A \sin c. \quad (6) \end{array}\right\}$$

When B is *middle* part, $(90^\circ - c)$, C are *adjacent* parts, $(90^\circ - A)$, $(90^\circ - b)$ are *opposite* parts,

then

$$\left.\begin{array}{ll} \sin B = \tan(90^\circ - c)\tan C & \text{or } \sin B = \cot c \tan C \quad (7) \\ \sin B = \cos(90^\circ - A)\cos(90^\circ - b) & \sin B = \sin A \sin b. \quad (8) \end{array}\right\}$$

When $(90^\circ - c)$ is *middle part*, B, $(90^\circ - A)$ are *adjacent* parts, C, $(90^\circ - b)$ are *opposite* parts,

then

$$\left.\begin{array}{ll} \sin(90^\circ - c) = -\tan B \tan(90^\circ - A) & \text{or } \cos c = -\tan B \cot A \quad (9) \\ \sin(90^\circ - c) = \cos C \cos(90^\circ - b) & \cos c = \cos C \sin b. \quad (10) \end{array}\right\}$$

Verification of the Ten Equations obtained by the Application of Napier's Rules to Quadrantal Triangles

68. These ten equations have been proved to be true in § 63. They may be verified as follows:

Write down the equation connecting the three parts in any formula with the quadrant, and substitute the values of the ratios of 90° which occur,

e.g. when a is a quadrant, prove $\cos A = -\cot b \cot c$.

The equation between A, a, b, c is

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c},$$

when $$a = 90^\circ,\ \cos a = 0,$$

and the equation reduces to

$$\cos A = -\cot b \cot c.$$

Similarly any of the other equations of § 67 may be verified.

69. Any of the equations obtained by the application of Napier's Rules to a right-angled spherical triangle, may be verified geometrically by using the method of § 73.

For example, when C is a right angle, prove $\sin a = \cot B \tan b$.

Constructing the figure as in § 73, we have

$$\sin a = \frac{RM}{OM} = \frac{RM . MP}{MP . OM} = \cot B \tan b.$$

If we have to verify any equation for a quadrantal triangle, we may proceed as follows.

Obtain the corresponding equation in the polar triangle which will be a right-angled triangle, and verify this. Obviously the equation for the quadrantal triangle from which it was derived must also be true.

For example, when c is a quadrant, prove $\cos a = -\tan B \cot C$.

The corresponding equation in the polar triangle, which will be right-angled at C′, is

$$\cos(180^\circ - A') = -\tan(180^\circ - b')\cot(180^\circ - c') \text{ or } \cos A' = \tan b' \cot c'.$$

Construct the figure as in § 73,

then $$\cos A' = \frac{SM}{SQ} = \frac{SM . OS}{OS . SQ} = \tan b' \cot c'.$$

Hence the formula $\cos a = -\tan B \cot C$ must also be true.

70. In equations (5), (6), (8), (9), (§§ 62 and 63) for the solution of both right-angled and quadrantal triangles, an angle has to be determined from its *sine;* and the question arises whether the *acute* angle found in the tables is the required angle, or whether we must take the *supplement* of this angle. No difficulty will arise, however, in making the proper selection if we remember that in a right-angled as also in a quadrantal triangle a side and the angle opposite it are of *like* affection, *i.e.* both less or both greater than 90°. This is proved in §§ 47, 48.

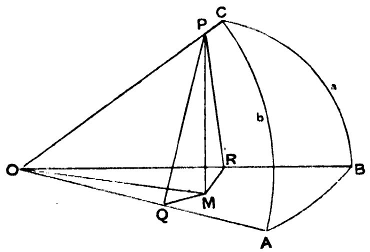
CHAPTER VI

GEOMETRICAL PROOFS OF FORMULÆ CONNECTING SIDES AND ANGLES OF A SPHERICAL TRIANGLE

71. Geometrical deduction of

1. The equation connecting two angles and the sides opposite them (A, B, a, b),

2. the equation connecting two angles and two sides all lying together (b, A, c, B or A, c, B, a).



Let ABC be a spherical triangle.

Take O, the centre of the sphere, and join OA, OB, OC.

In OC take any point P.

From P draw PM perpendicular to the plane AOB,

From M draw MQ perpendicular to OA,

From M draw MR perpendicular to OB;

join PQ, PR, OM.

$OP^2 = OM^2 + MP^2 = OQ^2 + QM^2 + PQ^2 - QM^2 = OQ^2 + PQ^2$ (Eu. i. 47)

Therefore PQO is a right angle (Euc. i. 48),

but MQO is a right angle by construction.

Hence PQM is the angle between the planes AOB, AOC, and therefore equals the spherical angle A (§ 13).

Similarly PRM is the spherical angle B.

Then

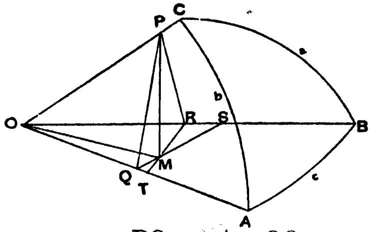
$$\frac{\sin A}{\sin B} = \frac{\dfrac{MP}{PQ}}{\dfrac{MP}{PR}} = \frac{PR}{PQ} = \frac{\dfrac{PR}{OP}}{\dfrac{PQ}{OP}} = \frac{\sin a}{\sin b}$$

or $$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b}.$$ This proves equation (1).

Similarly $$\frac{\sin A}{\sin a} = \frac{\sin C}{\sin c},$$

and therefore

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$



72. To prove the second equation.

With the same construction, as in § 71, and also produce QM to meet OB at S, and RM to meet OA at T.

then $RMS = QMT = c$,

$$QM = QS - MS,$$

or $$PQ \cos A = OQ \tan c - MR \sec c,$$

$\therefore$ $$\cos c \cos A = \frac{OQ}{PQ} \sin c - \frac{MR}{PQ}$$

$$= \cot b \sin c - \frac{MP \cot B}{MP \operatorname{cosec} A}$$

$$= \cos b \sin c - \cot B \sin A.$$

Again

$$MR = RT - MT,$$

or $$RP \cos B = OR \tan c - MQ \sec c,$$

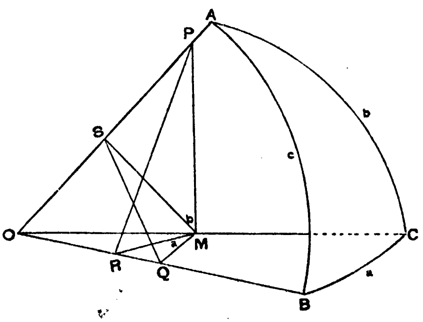
$\therefore$ $$\cos c \cos B = \frac{OR}{RP} \sin c - \frac{MQ}{RP}$$

$$= \cot a \sin c - \frac{MP \cot A}{MP \operatorname{cosec} B}$$

$$= \cot a \sin c - \cot A \sin B.$$

Geometrical Deduction of Napier's Rules for the Solution of Right-Angled Spherical Triangles

73. Let ABC be a spherical triangle right-angled at C. Take O the centre of the sphere.



Join OA, OB, OC.

In OC take any point M.

From M draw MP perpendicular to OC in the plane AOC.
" " MS " " OA " " "
" " MQ " " OC " " BOC.
" " MR " " OB " " "

Join PR, SQ,
because MP is perpendicular to OC and MS to OA,
therefore $PMS = AOC = b$.

Similarly $RMQ = BOC = a$.

And because the plane AOC is perpendicular to the plane BOC, and MP is drawn in the plane AOC perpendicular to OC the common section of these planes,
therefore MP is perpendicular to the plane BOC (Euc. xi. def. 3).

Hence PMR is a right angle ($=C$).

Similarly it may be shown that QMS is a right angle ($=C$).

Again, because
$OP^2 = OM^2 + MP^2 = (OR^2 + RM^2) + (PR^2 - RM^2) = OR^2 + PR^2$,
therefore ORP is a right angle (Euclid i. 48),
but ORM is a right angle by construction;
also OB is the common section of the planes AOB, BOC,
therefore PRM is the angle between these planes,
and consequently equals the spherical angle B (§ 13 (2)).
Similarly MSQ equals the spherical angle A (§ 13 (2)).

Hence

$$\begin{aligned} a &= QOM = ROM = RMQ, \\ b &= SOM = POM = SMP, \\ c &= POR = SOQ, \\ A &= QSM, \\ B &= PRM, \\ C &= PMR = SMQ. \end{aligned}$$

Then

$$\cos c = \cos a \cos b = \cot A \cot B,$$

for
$$\cos c = \frac{OR}{OP} = \frac{OR \cdot OM}{OM \cdot OP} = \cos a \cos b \qquad (1)$$

$$= \frac{RM \cdot SM}{QM \cdot MP} = \frac{SM \cdot RM}{QM \cdot MP} = \cot A \cot B \qquad (2),$$

$$\cos B = \tan a \cot c = \sin A \cos b,$$

for
$$\cos B = \frac{RM}{RP} = \frac{RM \cdot OR}{OR \cdot RP} = \tan a \cot c \qquad (3)$$

$$= \frac{QM \cdot OS}{OM \cdot QS} = \frac{QM \cdot OS}{QS \cdot OM} = \sin A \cos b \qquad (4),$$

$$\sin a = \cot B \tan b = \sin A \sin c,$$

for
$$\sin a = \frac{RM}{OM} = \frac{RM \cdot MP}{MP \cdot OM} = \cot B \tan b \qquad (5),$$

also $$\sin a = \frac{QM}{OQ} = \frac{QM \,.\, QS}{QS \,.\, OQ} = \sin A \sin c \qquad (6),$$

$$\sin b = \cot A \tan a = \sin B \sin c,$$

for $$\sin b = \frac{MS}{OM} = \frac{MS \,.\, MQ}{MQ \,.\, OM} = \cot A \tan a \qquad (7),$$

also $$\sin b = \frac{MP}{OP} = \frac{MP \,.\, PR}{PR \,.\, OP} = \sin B \sin c \qquad (8),$$

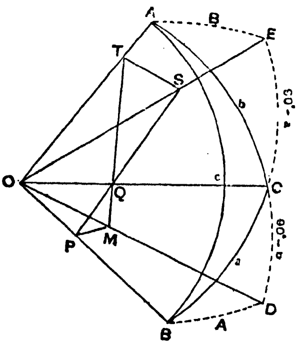
$$\cos A = \tan b \cot c = \sin B \cos a,$$

for $$\cos A = \frac{SM}{SQ} = \frac{SM \,.\, OS}{OS \,.\, SQ} = \tan b \cot c \qquad (9),$$

$$= \frac{MP \,.\, OR}{OM \,.\, RP} = \frac{MP \,.\, OR}{RP \,.\, OM} = \sin B \cos a \qquad (10).$$

74. *Note.*—If an equation involving sides *only* or else two sides and *one* angle has to be verified, we require to construct only one of the triangles PMR, SMQ. This is the case for equations (1), (3), (5), (6), (7), (8), (9). But if an equation involving *two* angles has to be verified, we require the whole figure. This is the case for the three equations (2), (4), (10).

Geometrical Deduction of Napier's Rules for the Solution of Quadrantal Triangles



75. Let ABC be a quadrantal triangle, AB being the quadrant.

Take O the centre of the sphere, and join OA, OB, OC.

Take BCE a quadrant, then BA being also a quadrant,

AE = B, and ∴ AOE = B.

Also take ACD a quadrant, then AB being also a quadrant,

BD = A, and ∴ BOD = A.

Through Q any point in OC draw in the plane BOE, PQS perpendicular to OC, meeting OB in P and OE in S.

Also through Q draw TQM in the plane AOD perpendicular to OC, meeting OA in T and OD in M.

Join TS, PM.

Then since OQP, OQM are right angles, OQ is perpendicular to the plane PQM.

Hence the plane COD passing through OQ is perpendicular to the plane PQM,

but the plane COD is perpendicular to the plane BOD (since BDA is a right angle).

Hence the planes PQM, BOD being both perpendicular to the plane COD, their common section PM is perpendicular to the plane COD.

Hence PMO, PMQ are right angles.

Similarly TSO, TSQ may be shown to be right angles.

Then SQT (being the angle between the planes EOC, AOC) $=$ ACE $= 180° - C$.

Also CE $= 90° - a$, $\therefore$ SOQ $= 90° - a$ and OSQ $= a$,

CD $= 90° - b$, $\therefore$ QOM $= 90° - b$ and QMO $= b$,

and SOP $= 90° =$ TOM.

Then

$$\cos C = -\cot a \cot b = -\cos A \cos B,$$

$$\text{for } \cos C = -\cos(180° - C) = -\frac{QS}{QT} = -\frac{QS \cdot OQ}{OQ \cdot QT} = -\cot a \cot b \quad (1)$$

$$= -\frac{OS \cdot OM}{OP \cdot OT} = -\frac{OS \cdot OM}{OT \cdot OP} = -\cos B \cos A \quad (2),$$

$$\cos b = -\tan A \cot C = \sin a \cos B,$$

$$\text{for } \cos b = \frac{QM}{OM} = \frac{QM \cdot MP}{MP \cdot OM} = \cot(180° - C)\tan A = -\cot C \tan A \quad (3)$$

$$\text{and } \cos b = \frac{OQ}{OT} = \frac{OQ \cdot OS}{OS \cdot OT} = \sin a \cos B \quad (4),$$

$$\sin A = \sin C \sin a = \cot b \tan B,$$

$$\text{for } \sin A = \frac{PM}{OP} = \frac{PM \cdot PQ}{PQ \cdot OP} = \sin(180° - C) \sin a = \sin C \sin a \quad (5)$$

$$= \frac{TS \cdot OQ}{QT \cdot OS} = \frac{TS \cdot OQ}{OS \cdot QT} = \tan B \cot b \quad (6),$$

$$\sin B = \sin C \sin b = \cot a \tan A,$$

$$\text{for } \sin B = \frac{TS}{OT} = \frac{TS \cdot TQ}{TQ \cdot OT} = \sin(180° - C) \sin b = \sin C \sin b \quad (7)$$

$$= \frac{PM \cdot OQ}{PQ \cdot OM} = \frac{PM \cdot OQ}{OM \cdot PQ} = \tan A \cot a \quad (8),$$

$$\cos a = -\tan B \cot C = \sin b \cos A,$$

$$\text{for } \cos a = \frac{QS}{OS} = \frac{QS \cdot ST}{ST \cdot OS} = \cot(180° - C) \tan B = -\cot C \tan B \quad (9)$$

$$\text{and } \cos a = \frac{OQ}{OP} = \frac{OQ \cdot OM}{OM \cdot OP} = \sin b \cos A \quad (10).$$

CHAPTER VII

SOLUTION OF OBLIQUE-ANGLED SPHERICAL TRIANGLES

76. The four general equations referred to in § 55, and established in §§ 56, 57, 58, 59, since they give the relations between any four parts of a spherical triangle, one of which must therefore be a side or else an angle, are sufficient for the complete solution of oblique-angled spherical triangles, just as they have been found sufficient for the solution of right-angled spherical triangles and quadrantal triangles. Only one of these equations, however, viz. $\frac{\sin A}{\sin a}=\frac{\sin B}{\sin b}=\frac{\sin C}{\sin c}$, is in a form suitable for use with logarithms.

It becomes necessary to reduce the remaining three equations to forms adapted for logarithmic computation, or, in other words, to prove rules for the solution of oblique-angled spherical triangles. We now proceed to do this.

77. The cases which arise are

CASE I. Given three sides, to find the three angles.

CASE II. Given two sides and the included angle, to find the remaining side and the remaining angles.

CASE III. Given three angles, to find the three sides.

CASE IV. Given two angles and the interjacent side, to find the remaining angle and the two remaining sides.

CASE V. Given two sides and an angle opposite one of these sides, to find the angle opposite the other side. Also to find the third side and third angle.

CASE VI. Given two angles and a side opposite one of these angles, to find the side opposite the other angle. Also to find the third side and third angle.

78. CASE I. Given three sides (a, b, c), to find the three angles (A, B, C).

$$\text{hav } A=\frac{\text{vers } A}{2}=\frac{1-\cos A}{2}=\frac{1-\dfrac{\cos a-\cos b\cos c}{\sin b\sin c}}{2}=\frac{\sin b\sin c-\cos a+\cos b\cos c}{2\sin b\sin c}$$

$$=\frac{\cos(b\sim c)-\cos a}{2\sin b\sin c}=\frac{2\sin\frac{1}{2}(a+b\sim c)\sin\frac{1}{2}(a-\overline{b\sim c})}{2\sin b\sin c}$$

$$=\text{cosec } b\text{ cosec } c\sqrt{\text{hav }(a+b\sim c)\text{ hav }(a-\overline{b\sim c})}.$$

Applying logs

$$\text{L hav A} = \text{L cosec } b + \text{L cosec } c + \tfrac{1}{2}\text{ L hav } (a + b \sim c) + \tfrac{1}{2}\text{ L hav } (a - \overline{b \sim c}) - 20.$$

Similarly

$$\text{L hav B} = \text{L cosec } a + \text{L cosec } c + \tfrac{1}{2}\text{ L hav } (b + a \sim c) + \tfrac{1}{2}\text{ L hav } (b - a \sim c) - 20,$$

and

$$\text{L hav C} = \text{L cosec } a + \text{L cosec } b + \tfrac{1}{2}\text{ L hav } (c + a \sim b) + \tfrac{1}{2}\text{ L hav } (c - a \sim b) - 20.$$

79. Case II. Given two sides and the included angle (b, c, A), to find the third side (a), and thence the two remaining angles (B, C).

$$\begin{aligned}\text{vers } a &= 1 - \cos a = 1 - (\cos b \cos c + \sin b \sin c \cos A)\\ &= 1 - \cos b \cos c - \sin b \sin c\,(1 - \text{vers } A)\\ &= 1 - \cos (b \sim c) + \sin b \sin c \text{ vers } A\\ &= \text{vers } (b \sim c) + \text{vers } \theta \qquad (1),\end{aligned}$$

where $\text{vers } \theta = \sin b \sin c \text{ vers } A$

and ∴ $\text{hav } \theta = \sin b \sin c \text{ hav } A \qquad (2).$

Applying logs to equation (2) we get

$$\text{L hav } \theta = \text{L sin } b + \text{L sin } c + \text{L hav A} - 20 \qquad (3).$$

Hence from (3) θ is known, and consequently from (1) a is known.

Now knowing the three sides a, b, c, we can find the angles B and C by Case I.

80. Case III. Given the three angles (A, B, C) to find the three sides (a, b, c).

This case may be solved by reducing the equation between a side and the three angles (see § 57) to a form adapted for logarithmic computation. This will be done later (see § 91). In practice, however, this case may be most conveniently solved by the aid of the polar triangle as follows

The three sides a', b', c' of the polar triangle are respectively supplements of the three angles A, B, C of the primitive triangle, and are therefore known.

With these three sides, by Case I., find the three angles A,′ B,′ C′ of the polar triangle. The supplements of these angles will be respectively the three sides a, b, c of the primitive triangle.

81. Case IV. Given two angles (B, C) and the interjacent side (a) to find the third angle (A) and the remaining sides (b and c).

The two sides b', c' and the included angle A′ of the polar triangle are respectively supplements of the two angles B, C and the interjacent side a of the primitive triangle.

By Case II. find the third side a' of the polar triangle. The supplement of this side a' will be the angle A of the primitive triangle.

Also knowing the three sides a', b', c' of the polar triangle, the angles B′, C′ may be determined by Case I.; their supplements will be respectively the sides b, c of the primitive triangle.

82. Case V. Given two sides and an angle opposite one of them, to find the angle opposite the other, *e.g.* given a, b, A to find B.

This is a case of the Rule of Sines and we have

$$\frac{\sin B}{\sin A} = \frac{\sin b}{\sin a}$$

or

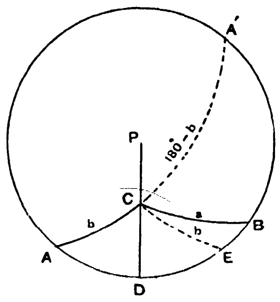
$$\sin B = \sin b \operatorname{cosec} a \sin A.$$

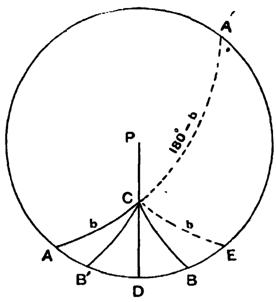
And applying logs

$$L \sin B = L \sin b + L \operatorname{cosec} a + L \sin A - 20.$$

We have now to determine whether the acute angle found in the tables is the required angle, or whether the obtuse angle which is the supplement of this angle is the angle sought, or whether we ought to take both values for the angle B.

By §§ 45, 46 we find that if a lie between b and $180° - b$ there will be only *one* triangle having the given parts, and B will be of *like* affection with b; but if a does *not* lie between b and $180° - b$ then there are *two* triangles having the given parts, and the *two* values of B will be *supplemental.*





From C the unknown angle draw CD perpendicular to the unknown side c.

When there is only *one* triangle this perpendicular falls *inside* the triangle if A and B are of *like* affection, but it will fall outside the triangle, beyond the obtuse angle and opposite the acute angle, if A and B are of *unlike* affection.

Then, by Napier's rules for the solution of right-angled triangles, find BCD and BD.

Find also ACD and AD or A′CD and A′D.

Then, when there is only *one* triangle,

$$C = ACD + BCD \text{ or } A'CD - BCD,$$
$$c = AD + BD \quad \text{ or } A'D - BD,$$

according as the perpendicular falls inside the triangle ABC or outside the triangle A′BC.

But when there are two triangles,

$$C = ACD \pm BCD \text{ or } A'CD \pm BCD,$$
$$c = AD \pm BD \text{ or } A'D \pm BD.$$

Note.—B will be greater or less than A according as b is greater or less than a, so that in practice if we find that the *acute* angle we first look out in the tables and its supplement, which will be *obtuse*, both fulfil this condition, we know that the triangle is *ambiguous.* But if only *one* of these angles fulfils this condition then there is only one triangle having the given parts.

83. Case VI. Given two angles and a side opposite one of them, to find the side opposite the other, *e.g.* given A, B, b to find a.

We have, by Rule of Sines,

$$\frac{\sin a}{\sin b} = \frac{\sin A}{\sin B}$$

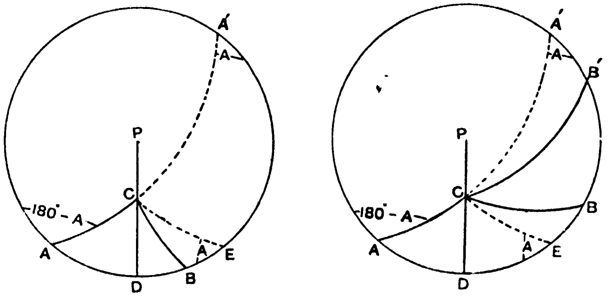
or
$$\sin a = \sin b \sin A \operatorname{cosec} B.$$

Or applying logs

$$L \sin a = L \sin b + L \sin A + L \operatorname{cosec} B - 20.$$

We have now to determine, as in Case V., whether the acute angle found in the tables is the required angle, or whether we must take the obtuse angle, which is the supplement of this, or whether we must take both values for the side a.

By §§ 45, 46 we find that if B lie between A and $180° - A$ there will be only *one* triangle having the given parts, and a will be of *like* affection with A; but if B does not lie between A and $180° - A$ there will be *two* triangles having the given parts, and the two values of a will be supplemental.



From C the unknown angle draw CD perpendicular to the unknown side c.

How this perpendicular will fall is pointed out in Case V. § 82.

Then when there is only one triangle, by Napier's rules for the solution of right-angled triangles,

find ACD, BCD, also AD and BD,

then
$$C = ACD + BCD \text{ or } A'CD - BCD,$$
$$c = AD + BD \quad \text{ or } A'D - BD,$$

according as the perpendicular falls inside the triangle ABC or outside the triangle A′BC.

But when there are two triangles,
since CB′ is the supplement of CB,
therefore B′CD is the supplement of BCD (§ 49 (9)),
and B′D is the supplement of BD.

Hence $C = ACD + BCD$ or $ACD + B'CD = ACD + (180^\circ - BCD)$,
$c = AD + BD$ or $AD + B'D = AD + (180^\circ - BD)$.

Note.—The same remark may be made as in Case V., the ambiguity or otherwise will declare itself when we determine the side a.

Note as to the use of the Rule of Sines.—It would appear that when in a spherical triangle three sides and one angle are known, or else three angles and one side, the rule of sines might be advantageously used to complete the solution of the triangle, since only four logarithms will be required in place of six in order to determine the other two parts.

It should be noted, however, that the parts involved in the rule of sines may give rise to ambiguity, and that this cannot be removed. Hence it is best to avoid the rule of sines for this purpose, as the time spent in ascertaining whether we may expect ambiguity or not, will be lost should it turn out that such ambiguity will arise.

The following are examples of this difficulty:—I. 1, 2, 3, 4, 5, 9; II. 5, 6, 10; III. 4, 6, 7, 9, 10; IV. 1, 3, 4, 6, 7, 9, 10; IX. 2, 5, 12, 13, 17.

Further, it may be observed that should the angle we are seeking be close to 90°, it cannot be determined accurately from its L sine; as examples II. 10, find A; III. 7, find c.

CHAPTER VIII

SOLUTION OF OBLIQUE-ANGLED SPHERICAL TRIANGLES—ALTERNATIVE METHODS

84. The methods already given are sufficient for the solution of all cases of a spherical triangle. Some of them, however, necessitate the use of the L Haversine or Tabular Versed sine Tables which are not always available. In Case II. the third side, and in Cases V. and VI. the third side and third angle, may be found by the aid of a subsidiary angle without the use of Haversines or Versed sines or of Napier's rules for the solution of right-angled triangles. We proceed to give these methods.

The following notation is used in some of the proofs.

If
$$s = \tfrac{1}{2}(a + b + c),$$
then
$$\begin{cases} s - a = \tfrac{1}{2}(b + c - a), \\ s - b = \tfrac{1}{2}(a - b + c), \\ s - c = \tfrac{1}{2}(a + b - c). \end{cases}$$

85. CASE I. Given the three sides (a, b, c), to find the three angles (A, B, C).

$$(1)\ \sin\frac{A}{2} = \sqrt{\frac{1 - \cos A}{2}} = \sqrt{\frac{1 - \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}}{2}}$$

$$= \sqrt{\frac{\sin b \sin c - \cos a + \cos b \cos c}{2 \sin b \sin c}} = \sqrt{\frac{\cos (b \sim c) - \cos a}{2 \sin b \sin c}}$$

$$= \sqrt{\frac{2 \sin \frac{1}{2}(a + b \sim c) \sin \frac{1}{2}(a - b \sim c)}{2 \sin b \sin c}} = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}}.$$

Applying logs

$$\text{L} \sin \frac{A}{2} = \tfrac{1}{2}\{\text{L} \sin (s - b) + \text{L} \sin (s - c) - \text{L} \sin b - \text{L} \sin c\} + 10$$

or

$$\text{L} \sin \frac{A}{2} = \tfrac{1}{2}\{\text{L} \sin (s - b) + \text{L} \sin (s - c) + \text{L} \operatorname{cosec} b + \text{L} \operatorname{cosec} c\} - 10$$

$$(2)\ \cos\frac{A}{2}=\sqrt{\frac{1+\cos A}{2}}=\sqrt{\frac{1+\dfrac{\cos a-\cos b\cos c}{\sin b\sin c}}{2}}$$

$$=\sqrt{\frac{\sin b\sin c+\cos a-\cos b\cos c}{2\sin b\sin c}}=\sqrt{\frac{\cos a-\cos(b+c)}{2\sin b\sin c}}$$

$$=\sqrt{\frac{2\sin\frac{1}{2}(b+c+a)\sin\frac{1}{2}(b+c-a)}{2\sin b\sin c}}=\sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}}.$$

Applying logs

$$L\cos\frac{A}{2}=\tfrac{1}{2}\{L\sin s+L\sin(s-a)-L\sin b-L\sin c\}+10$$

or

$$L\cos\frac{A}{2}=\tfrac{1}{2}\{L\sin s+L\sin(s-a)+L\operatorname{cosec} b+L\operatorname{cosec} c\}-10.$$

$$(3)\ \tan\frac{A}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=\sqrt{\frac{1-\cos A}{1+\cos A}}=\sqrt{\frac{1-\dfrac{\cos a-\cos b\cos c}{\sin b\sin c}}{1+\dfrac{\cos a-\cos b\cos c}{\sin b\sin c}}}$$

$$=\sqrt{\frac{\sin b\sin c-\cos a+\cos b\cos c}{\sin b\sin c+\cos a-\cos b\cos c}}=\sqrt{\frac{\cos(b\sim c)-\cos a}{\cos a-\cos(b+c)}}$$

$$=\sqrt{\frac{2\sin\frac{1}{2}(a+b\sim c)\sin\frac{1}{2}(a-b\sim c)}{2\sin\frac{1}{2}(b+c+a)\sin\frac{1}{2}(b+c-a)}}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}.$$

Applying logs

$$L\tan\frac{A}{2}=\tfrac{1}{2}\{L\sin(s-b)+L\sin(s-c)-L\sin s-L\sin(s-a)\}+10.$$

$$(4)\ \tan\frac{A}{2}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}=\frac{1}{\sin(s-a)}\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}$$

$$=\frac{X}{\sin(s-a)},$$

where $X=\sqrt{\dfrac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}$.

Applying logs

$$L\tan\frac{A}{2}=10+\tfrac{1}{2}\{L\sin(s-a)+L\sin(s-b)+L\sin(s-c)-L\sin s\}-L\sin(s-a)$$

$$=10+\log X-L\sin(s-a).$$

Note.—If all three angles have to be calculated, equation (4) is the best of these methods to use. Four openings in the tables give all the necessary logs, and also after $10+\log X$ has been found the successive subtraction of $L\sin(s-a)$, $L\sin(s-b)$, $L\sin(s-c)$ will give us the L tangents of $\frac{A}{2}$, $\frac{B}{2}$, $\frac{C}{2}$ respectively.

The following methods are good illustrations of the use of subsidiary angles in calculations.

86. Case II. Given two sides and the included angle (a, b, C), to find the third side (c) and the remaining angles (A, B).

The equation connecting a, b, c, C is

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$
$$= \cos b\{\cos a + \sin a \tan b \cos C\}.$$

Assume $$\tan \theta = \tan b \cos C \qquad (1),$$
then

$$\cos c = \cos b\{\cos a + \tan \theta \sin a\} = \frac{\cos b \cos (a \sim \theta)}{\cos \theta}$$

or $$\cos c : \cos b :: \cos (a \sim \theta) : \cos \theta \qquad (2).$$

Applying logs to equations (1) and (2), we get

$$\left.\begin{aligned} L \tan \theta &= L \tan b + L \cos C - 10 \\ \text{and} \quad L \cos c &= L \cos b + L \cos (a \sim \theta) + L \sec \theta - 20 \end{aligned}\right\} \qquad (3).$$

Then, by one of the methods of § 85, knowing the three sides, find the two angles A, B.

87. Case III. Given three angles (A, B, C), to find the three sides (a, b, c).

This case may be solved by the aid of the polar triangle as described in § 80, using one of the methods given in § 85.

88. Case IV. Given two angles and the interjacent side (B, C, a), to find the third angle and the remaining sides (A, b, c).

This case may also be solved by the aid of the polar triangle as described in § 81, using the method of § 86 and then one of the methods given in § 85.

89. Case V. Given two sides and an angle opposite one of the sides (A, b, a), to find the third side (c) and angle opposite it (C).

To find c.

The equation connecting a, b, c, A is

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$
$$= \cos b\{\cos c + \sin c \tan b \cos A\}.$$

Assume $$\tan \phi = \tan b \cos A \qquad (1),$$
then

$$\cos a = \cos b\{\cos c + \sin c \tan \phi\} = \frac{\cos b \cos (c \sim \phi)}{\cos \phi}$$

or $$\cos (c \sim \phi) : \cos \phi :: \cos a : \cos b \qquad (2).$$

Applying logs to equations (1) and (2), we get

$$\left.\begin{aligned} L \tan \phi &= L \tan b + L \cos A - 10 \\ \text{and} \quad L \cos (c \sim \phi) &= L \cos \phi + L \cos a + L \sec b - 20 \end{aligned}\right\} \qquad (3).$$

To find C.

The equation connecting A, b, C, a is

$$\cos b \cos C = \cot a \sin b - \cot A \sin C,$$

$$\therefore \quad \cot a \sin b = \cos b \cos C + \cot A \sin C$$
$$= \cos b\{\cos C + \cot A \sec b \sin C\}.$$

Assume $\tan\theta = \cot A \sec b$ (1),

then

$$\cot a \tan b = \cos C + \tan\theta \sin C = \frac{\cos(C \sim \theta)}{\cos\theta}$$

or $\cos(C \sim \theta) : \cos\theta :: \cot a : \cot b$ (2).

Applying logs to equations (1) and (2), we get

$$\left.\begin{aligned} L\tan\theta &= L\cot A + L\sec b - 10 \\ \text{and} \quad L\cos(C \sim \theta) &= L\cos\theta + L\cot a + L\tan b - 20 \end{aligned}\right\} \quad (3).$$

Note.—The angle B may be found by Rule of Sines, as in § 82.

90. CASE VI. Given two angles and a side opposite one of them (A, B, b), to find the third angle (C) and the side opposite it (c).

To find C.

The equation connecting A, B, C, b is

$$\cos b = \frac{\cos B + \cos A \cos C}{\sin A \sin C}$$

or $-\cos B = \cos A \cos C - \cos b \sin A \sin C,$

$$\frac{-\cos B}{\cos A} = \cos C - \cos b \tan A \sin C.$$

Assume $\tan\theta = -\cos b \tan A$ (1),

then

$$\frac{-\cos B}{\cos A} = \frac{\cos C \cos\theta + \sin C \sin\theta}{\cos\theta} = \frac{\cos(C \sim \theta)}{\cos\theta}$$

or $\cos(C \sim \theta) : \cos\theta :: -\cos B : \cos A$ (2).

Applying logs to equations (1) and (2), we get

$$\left.\begin{aligned} L\tan\theta &= L\cos b + L\tan A - 10 \\ \text{and} \quad L\cos(C \sim \theta) &= L\cos\theta + L\cos B + L\sec A - 20 \end{aligned}\right\} \quad (3).$$

To find c.

The equation connecting b, A, c, B is

$$\cos A \cos c = \cot b \sin c - \cot B \sin A,$$
$$\cot B \sin A = \cot b \sin c - \cos A \cos c,$$
$$\cot B \tan A = \cot b \sec A \sin c - \cos c.$$

Assume $-\tan\phi = \cot b \sec A$ (1),

then

$$\frac{\tan A}{\tan B} = -\frac{\{\sin c \sin\phi + \cos c \cos\phi\}}{\cos\phi} = -\frac{\cos(c \sim \phi)}{\cos\phi}$$

or $\cos(c \sim \phi) : \cos\phi :: -\tan A : \tan B$ (2).

Applying logs to equations (1) and (2), we get

$$\left.\begin{aligned} L\tan\phi &= L\cot b + L\sec A - 10 \\ \text{and} \quad L\cos(c \sim \phi) &= L\cos\phi + L\tan A + L\cot B - 20 \end{aligned}\right\} \quad (3).$$

Note.—The side a may be found by Rule of Sines, as in § 83.

CHAPTER IX

SOLUTION OF OBLIQUE-ANGLED SPHERICAL TRIANGLES—SECOND GROUP OF ALTERNATIVE METHODS

91. **Case III.** Given the three angles A, B, C, to find the three sides a, b, c.

Let $A+B+C=180^\circ+E$, then $\frac{A+B+C}{2}=90^\circ+\frac{E}{2}$ and

$$\frac{B+C-A}{2}=90^\circ-\left(A-\frac{E}{2}\right),\ \frac{A+C-B}{2}=90^\circ-\left(B-\frac{E}{2}\right),\ \frac{A+B-C}{2}=90^\circ-\left(C-\frac{E}{2}\right)$$

$$\text{hav}\,a=\frac{\text{vers}\,a}{2}=\frac{1-\cos a}{2}=\frac{1-\dfrac{\cos A+\cos B\cos C}{\sin B\sin C}}{2}=\frac{\sin B\sin C-\cos A-\cos B\cos C}{2\sin B\sin C}$$

$$=\frac{-\cos A-\cos(B+C)}{2\sin B\sin C}=\frac{-2\cos\dfrac{A+B+C}{2}\cos\dfrac{B+C-A}{2}}{2\sin B\sin C}$$

$$=\frac{-\cos\left(90^\circ+\dfrac{E}{2}\right)\cos\left\{90^\circ-\left(A-\dfrac{E}{2}\right)\right\}}{\sin B\sin C}=\frac{\sin\dfrac{E}{2}\sin\dfrac{2A-E}{2}}{\sin B\sin C}$$

$$=\text{cosec}\,B\,\text{cosec}\,C\,\sqrt{\text{hav}\,E\,\text{hav}\,(2A-E)} \qquad (1).$$

$$\sin\frac{a}{2}=\sqrt{\frac{1-\cos a}{2}}=\sqrt{\frac{1-\dfrac{\cos A+\cos B\cos C}{\sin B\sin C}}{2}}$$

$$=\sqrt{\frac{\sin B\sin C-\cos A-\cos B\cos C}{2\sin B\sin C}}=\sqrt{\frac{-\cos A-\cos(B+C)}{2\sin B\sin C}}$$

$$=\sqrt{\frac{-2\cos\frac{1}{2}(A+B+C)\cos\frac{1}{2}(B+C-A)}{2\sin B\sin C}}$$

$$=\sqrt{\frac{-\cos\left(90^\circ+\dfrac{E}{2}\right)\cos\left\{90^\circ-\left(A-\dfrac{E}{2}\right)\right\}}{\sin B\sin C}}=\sqrt{\frac{\sin\dfrac{E}{2}\sin\left(A-\dfrac{E}{2}\right)}{\sin B\sin C}} \qquad (2).$$

$$\cos\frac{a}{2}=\sqrt{\frac{1+\cos a}{2}}=\sqrt{\frac{1+\dfrac{\cos A+\cos B\cos C}{\sin B\sin C}}{2}}$$

$$=\sqrt{\frac{\sin B\sin C+\cos A+\cos B\cos C}{2\sin B\sin C}}=\sqrt{\frac{\cos A+\cos(B\sim C)}{2\sin B\sin C}}$$

$$=\sqrt{\frac{2\cos\frac{1}{2}(A-B+C)\cos\frac{1}{2}(A+B-C)}{2\sin B\sin C}}$$

$$=\sqrt{\frac{\cos\left\{90^\circ-\left(B-\frac{E}{2}\right)\right\}\cos\left\{90^\circ-\left(C-\frac{E}{2}\right)\right\}}{\sin B\sin C}}=\sqrt{\frac{\sin\left(B-\frac{E}{2}\right)\sin\left(C-\frac{E}{2}\right)}{\sin B\sin C}} \quad (3).$$

$$\tan\frac{a}{2}=\frac{\sin\frac{a}{2}}{\cos\frac{a}{2}}=\sqrt{\frac{\sin\frac{E}{2}\sin\left(A-\frac{E}{2}\right)}{\sin\left(B-\frac{E}{2}\right)\sin\left(C-\frac{E}{2}\right)}} \quad (4).$$

Equation (4) can also be written

$$\tan\frac{a}{2}=\sin\left(A-\frac{E}{2}\right)\sqrt{\frac{\sin\frac{E}{2}}{\sin\left(A-\frac{E}{2}\right)\sin\left(B-\frac{E}{2}\right)\sin\left(C-\frac{E}{2}\right)}}$$

$$=\sin\left(A-\frac{E}{2}\right)\times X \quad (5).$$

where X is written for the expression

$$\sqrt{\frac{\sin\frac{E}{2}}{\sin\left(A-\frac{E}{2}\right)\sin\left(B-\frac{E}{2}\right)\sin\left(C-\frac{E}{2}\right)}}.$$

92. Since any two sides of a spherical triangle are greater than the third,

$\therefore$ in the polar triangle $b'+c'>a'$,

$\therefore$ $180^\circ-B+180^\circ-C>180^\circ-A$

or $B+C-A<180^\circ$,

$\therefore$ $B+C+A-180^\circ<2A$

or $E<2A$

and $\frac{E}{2}<A.$

Hence $\frac{E}{2}$, $\left(A-\frac{E}{2}\right)$, $\left(B-\frac{E}{2}\right)$, $\left(C-\frac{E}{2}\right)$ are all positive and less than 180°, so that their sines are all positive,

also $\operatorname{hav} E=\sin^2\frac{E}{2}$, $\operatorname{hav}(2A-E)=\sin^2\left(\frac{2A-E}{2}\right)=\sin^2\left(A-\frac{E}{2}\right)$,

so that these ratios are positive.

Hence formulæ (1) to (5) are all real.

Applying logs to equations (1), (2), (3), (4), (5), we get

$$L\operatorname{hav} a=L\operatorname{cosec} B+L\operatorname{cosec} C+\tfrac{1}{2}L\operatorname{hav} E+\tfrac{1}{2}L\operatorname{hav}(2A-E)-20 \quad (1),$$

$$L\sin\frac{a}{2}=\frac{1}{2}\left\{L\sin\frac{E}{2}+L\sin\left(A-\frac{E}{2}\right)+L\operatorname{cosec} B+L\operatorname{cosec} C\right\}-10 \quad (2),$$

$$L\cos\frac{a}{2}=\frac{1}{2}\left\{L\sin\left(B-\frac{E}{2}\right)+L\sin\left(C-\frac{E}{2}\right)+L\operatorname{cosec} B+L\operatorname{cosec} C\right\}-10 \quad (3),$$

$$L\tan\frac{a}{2}=\frac{1}{2}\left\{L\sin\frac{E}{2}+L\sin\left(A-\frac{E}{2}\right)-L\sin\left(B-\frac{E}{2}\right)-L\sin\left(C-\frac{E}{2}\right)\right\}+10 \quad (4),$$

$$L\tan\frac{a}{2}=\frac{1}{2}\left\{L\sin\frac{E}{2}-L\sin\left(A-\frac{E}{2}\right)-L\sin\left(B-\frac{E}{2}\right)-L\sin\left(C-\frac{E}{2}\right)\right\}$$
$$+L\sin\left(A-\frac{E}{2}\right)+10$$
$$=10+\log X+L\sin\left(A-\frac{E}{2}\right) \quad (5).$$

Note.—If all three sides have to be found the last equation is the best of these methods to use. Four openings in the tables give all the necessary logs, and also after $10+\log X$ has been calculated the successive addition of $L\sin\left(A-\frac{E}{2}\right)$, $L\sin\left(B-\frac{E}{2}\right)$, $L\sin\left(C-\frac{E}{2}\right)$ will give us the L tangents of $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$ respectively.

93. In § 85 formulæ for $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ were obtained in terms of functions of the sides,

viz.
$$\sin\frac{A}{2}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}},$$
$$\cos\frac{A}{2}=\sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}},$$
with similar formulæ for $\frac{B}{2}, \frac{C}{2}$.

By the aid of these formulæ Gauss's Theorems and Napier's Analogies may be demonstrated. It will then be found that these Theorems, but more especially Napier's Analogies, can be applied to the solution of several cases of spherical triangles.

94. Demonstration of Gauss's Theorems and Napier's Analogies

$$\sin\frac{A}{2}\cos\frac{B}{2}=\sqrt{\frac{\sin(s-b)\sin(s-c)\sin s\sin(s-b)}{\sin b\sin c\sin a\sin c}}=\frac{\sin(s-b)}{\sin c}\cos\frac{C}{2} \quad (1),$$

$$\cos\frac{A}{2}\sin\frac{B}{2}=\sqrt{\frac{\sin s\sin(s-a)\sin(s-a)\sin(s-c)}{\sin b\sin c\sin a\sin c}}=\frac{\sin(s-a)}{\sin c}\cos\frac{C}{2} \quad (2),$$

by addition and subtraction

$$\sin\frac{A+B}{2}=\frac{\sin(s-a)+\sin(s-b)}{\sin c}\cos\frac{C}{2}=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{c}{2}}\cos\frac{C}{2} \quad (3),$$

$$\sin\frac{A-B}{2}=\frac{\sin(s-b)-\sin(s-a)}{\sin c}\cos\frac{C}{2}=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{c}{2}}\cos\frac{C}{2} \quad (4).$$

Formulæ (3) and (4) are called Gauss's Theorems, but they are really due to Delambre.

Again

$$\cos\frac{A}{2}\cos\frac{B}{2}=\sqrt{\frac{\sin s\sin(s-a)\sin s\sin(s-b)}{\sin b\sin c\sin a\sin c}}=\frac{\sin s}{\sin c}\sin\frac{C}{2} \qquad (5),$$

$$\sin\frac{A}{2}\sin\frac{B}{2}=\sqrt{\frac{\sin(s-b)\sin(s-c)\sin(s-a)\sin(s-c)}{\sin b\sin c\sin a\sin c}}=\frac{\sin(s-c)}{\sin c}\sin\frac{C}{2} \qquad (6),$$

by subtraction and addition

$$\cos\frac{A+B}{2}=\frac{\sin s-\sin(s-c)}{\sin c}\sin\frac{C}{2}=\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{c}{2}}\sin\frac{C}{2} \qquad (7),$$

$$\cos\frac{A-B}{2}=\frac{\sin s+\sin(s-c)}{\sin c}\sin\frac{C}{2}=\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{c}{2}}\sin\frac{C}{2} \qquad (8).$$

Dividing (3) by (7) and (4) by (8),

$$\tan\frac{A+B}{2}=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\frac{C}{2} \qquad (9),$$

$$\tan\frac{A-B}{2}=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\frac{C}{2} \qquad (10).$$

Dividing (4) by (3) and (8) by (7),

$$\frac{\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}}=\frac{\tan\frac{1}{2}(a-b)}{\tan\frac{1}{2}c} \qquad (11),$$

$$\frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}=\frac{\tan\frac{1}{2}(a+b)}{\tan\frac{c}{2}} \qquad (12).$$

Formulæ (9), (10), (11), (12) are called Napier's Analogies.

Note.—$\frac{1}{2}(A+B)$ and $\frac{1}{2}(a+b)$ are of like affection, that is both greater or both less than a right angle.

For, in Napier's Analogy, $\tan\frac{A+B}{2}=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\frac{C}{2}$,

$\cos\frac{1}{2}(a-b)$ and $\cot\frac{C}{2}$ are necessarily positive quantities.

Hence the equation shows that $\tan\frac{A+B}{2}$ and $\cos\frac{1}{2}(a+b)$ must be of the same sign; thus $\frac{A+B}{2}$ and $\frac{a+b}{2}$ are either both less or both greater than a right angle.

95. CASE II. Given two sides and the included angle (a, b, C), to find (A, B, c).

To find A, B.

By Napier's Analogies,

$$\tan \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \frac{C}{2},$$

$$\tan \tfrac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cot \frac{C}{2},$$

then

$$A = \tfrac{1}{2}(A+B) + \tfrac{1}{2}(A-B),$$
$$B = \tfrac{1}{2}(A+B) - \tfrac{1}{2}(A-B).$$

To find c.

By Napier's Analogies,

$$\tan \frac{c}{2} = \frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)} \tan \tfrac{1}{2}(a+b).$$

96. CASE IV. Given two angles and the interjacent side (A, c, B), to find (a, b, C).

To find a, b.

By Napier's Analogies,

$$\tan \tfrac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \tan \frac{c}{2},$$

$$\tan \tfrac{1}{2}(a-b) = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \tan \frac{c}{2},$$

then

$$a = \tfrac{1}{2}(a+b) + \tfrac{1}{2}(a-b),$$
$$b = \tfrac{1}{2}(a+b) - \tfrac{1}{2}(a-b).$$

To find C.

By Napier's Analogies,

$$\cot \frac{C}{2} = \frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}(a-b)} \tan \tfrac{1}{2}(A+B).$$

97. CASE V. Given two sides and an angle opposite one of them (a, b, A), to find (B, C, c).

To find B.

By Rule of Sines,

$$\sin B = \frac{\sin b}{\sin a} \sin A.$$

To find C, c.

By Napier's Analogies,

$$\tan \frac{C}{2} = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \tfrac{1}{2}(A+B),$$

$$\tan \frac{c}{2} = \frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)} \tan \tfrac{1}{2}(a+b).$$

98. CASE VI. Given two angles and a side opposite one of them (A, B, a), to find (b, C, c).

To find b.

By Rule of Sines,

$$\sin b = \frac{\sin B}{\sin A} \sin a.$$

To find C, c.

By Napier's Analogies,

$$\tan \frac{C}{2} = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \frac{1}{2}(A+B),$$

$$\tan \frac{c}{2} = \frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)} \tan \frac{1}{2}(a+b).$$

CHAPTER X

PRACTICAL SOLUTION OF RIGHT-ANGLED SPHERICAL TRIANGLES AND QUADRANTAL TRIANGLES

I. Solution of right-angled Spherical Triangles by Napier's Rules

99. (1) Given $a=37^\circ\ 48'$, $b=59^\circ\ 44'\ 15''$, $C=90^\circ$, find the other parts.

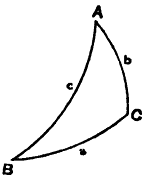
The circular parts are $(90^\circ-c)$, $(90^\circ-A)$, $(90^\circ-B)$, a, b.

To find c.

$(90^\circ-c)$ is the middle part, a and b are the opposite parts.

By Rule 2,

$$\begin{aligned} \cos c &= \cos b \cos a \\ L \cos b &= 9{\cdot}702398 \\ L \cos a &= 9{\cdot}897712 \\ \hline L \cos c &= \bar{1}9{\cdot}600110 \\ \hline c &= 66^\circ\ 32' \end{aligned}$$



To find A.

b is middle part, $(90^\circ-A)$ and a are adjacent parts.

By Rule 1,

$$\sin b = \cot A \tan a$$

or

$$\begin{aligned} \cot A &= \sin b\ \cot a \\ L \sin b &= 9{\cdot}936376 \\ L \cot a &= 10{\cdot}110318 \\ \hline L \cot A &= 10{\cdot}046694 \\ \hline A &= 41^\circ 55' 30'' \end{aligned}$$

To find B.

a is middle part, $(90^\circ-B)$ and b are adjacent parts.

By Rule 1, $\sin a = \cot B \tan b$

or $\cot B = \sin a \cot b$

$$\begin{aligned} L \sin a &= 9{\cdot}787395 \\ L \cot b &= 9{\cdot}766022 \\ \hline L \cot B &= 9{\cdot}553417 \\ \hline B &= 70^\circ 19' 15'' \end{aligned}$$

100. (2) Given $A = 55^\circ\ 32'\ 45''$, $C = 90^\circ$, $c = 98^\circ\ 14'\ 30''$, find the other parts.

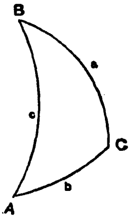
Here c being greater than 90° its sine and cosecant will alone be positive, all its other ratios will be negative. In such cases it is necessary to give to the ratios of the known parts, as they occur in the formulæ, their proper signs, and thus we are able to determine the sign of the ratio of the angle we are seeking. If this is positive the angle sought is the acute angle found in the tables, but if negative, the supplement of this angle is the required part. Further, it must be remembered that if a part has to be determined from its sine, a side and the opposite angle in a right-angled triangle will be of *like* affection. This does away at once with any ambiguity.

The circular parts are a, b, $(90^\circ - c)$, $(90^\circ - A)$, $(90^\circ - B)$.

To find a.

a is middle part, $(90^\circ - A)$ and $(90^\circ - c)$ are opposite parts.

By Rule 2, $\sin a = \sin A \sin c$



$$\begin{aligned} L \sin A &= 9{\cdot}916232 \\ L \sin c &= 9{\cdot}995491 \\ \hline L \sin a &= 9{\cdot}911723 \\ \hline a &= 54^\circ 41' 30'' \end{aligned}$$

for a must be of *like* affection with A.

To find B.

$(90^\circ - c)$ is middle part, $(90^\circ - A)$ and $(90^\circ - B)$ are adjacent parts.

By Rule 1,

$$\overset{-}{\cos c} = \overset{+}{\cot A}\ \overset{-}{\cot B}$$

or $\cot B = \cos c \tan A$

$$\begin{aligned} L \cos c &= 9{\cdot}156394 \\ L \tan A &= 10{\cdot}163610 \\ \hline L \cot B &= 9{\cdot}320004 \\ \hline & 78^\circ\ 12'\ 0'' \\ & 180 \\ \hline B &= 101^\circ\ 48'\ 0'', \text{ since cot B is negative.} \end{aligned}$$

To find b.

$(90^\circ - A)$ is middle part, $(90^\circ - c)$ and b are adjacent parts.

By Rule 1,

$$\overset{+}{\cos A} = \overset{-}{\cot c}\ \overset{-}{\tan b}$$

or

$$\tan b = \cos A \tan c$$

$$L \cos A = 9{\cdot}752622$$
$$L \tan c = 10{\cdot}839098$$
$$L \tan b = 10{\cdot}591720$$
$$75^\circ\ 38'\ 30''$$
$$180$$
$$b = 104^\circ\ 21'\ 30'', \text{ since } \tan b \text{ is negative.}$$

101. (3) Given $A = 110^\circ\ 20'$, $a = 120^\circ\ 15'$, $B = 90^\circ$, find the other parts.

This case is ambiguous (see § 49 (9)), and with the exception of A and a, which are common to both triangles, the other parts will be supplemental. Thus c and $180^\circ - c$, b and $180^\circ - b$ are corresponding sides, whilst C and $180^\circ - C$ are corresponding angles. This is also evident from the figure.

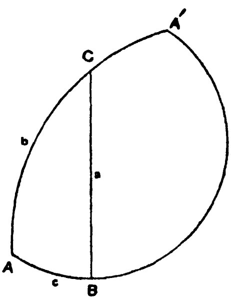
The circular parts are a, c, $(180^\circ - b)$, $(180^\circ - A)$, $(180^\circ - C)$.

To find c.

c is middle part, a and $(180^\circ - A)$ are adjacent parts.

By Rule 1,

$$\sin c = \tan a \cot A$$
$$L \tan a = 10{\cdot}234195$$
$$L \cot A = 9{\cdot}568873$$
$$L \sin c = 9{\cdot}803068$$
$$c = 39^\circ\ 27'\ 0''$$
$$180$$
$$\text{or } = 140^\circ\ 33'\ 0''$$



To find C.

$(90^\circ - A)$ is middle part, $(90^\circ - C)$ and a are opposite parts.

By Rule 2,

$$\cos A = \sin C \cos a$$

or

$$\sin C = \cos A \sec a$$

$$\begin{aligned} L \cos A &= 9{\cdot}540931 \\ L \sec a &= 10{\cdot}297764 \\ \hline L \sin C &= 9{\cdot}838695 \\ \hline C &= 43^\circ 36' 45'' \\ & \quad 180 \\ \hline \text{or} &= 136^\circ 23' 15'' \end{aligned}$$

To find b.

a is middle part, $(90^\circ - b)$ and $(90^\circ - A)$ are opposite parts.

By Rule 2,

$$\sin a = \sin b \sin A$$

or

$$\sin b = \sin a \operatorname{cosec} A$$

$$\begin{aligned} L \sin a &= 9{\cdot}936431 \\ L \operatorname{cosec} A &= 10{\cdot}027942 \\ \hline L \sin b &= 9{\cdot}964373 \\ \hline b &= 67^\circ \ 6' \ 30'' \\ & \quad 180 \\ \hline \text{or} &= 112^\circ \ 53' \ 30'' \end{aligned}$$

Note.—There is evidently no necessity in this case to take any notice of the signs of the ratios of A and a.

II. Solution of Quadrantal Triangles by Napier's Rules

102. (1) Given $A = 35^\circ \ 40' \ 15''$, $B = 36^\circ \ 10'$, $c = 90^\circ$.

The circular parts are

$$A, B, (90^\circ - C), (90^\circ - a), (90^\circ - b).$$

To find a.

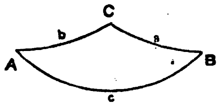
B is middle part, $(90^\circ - a)$ and A are adjacent parts.

By Rule 1,

$$\sin B = \cot a \tan A$$

or

$$\cot a = \sin B \cot A$$



$$\begin{aligned} L \sin B &= 9{\cdot}770952 \\ L \cot A &= 10{\cdot}143996 \\ \hline L \cot a &= 9{\cdot}914948 \\ \hline a &= 50^\circ \ 34' \ 30'' \end{aligned}$$

To find C.

$(90^\circ - C)$ is middle part, A and B are opposite parts.

By Rules 2 and 3,

$$\cos C = -\cos A \cos B$$

$$\begin{aligned} L \cos A &= 9{\cdot}909759 \\ L \cos B &= 9{\cdot}907037 \\ \hline L \cos C &= 9{\cdot}816796 \\ \hline & 49^\circ\ 1'\ 0'' \\ & 180 \\ \hline C &= 130^\circ\ 59'\ 0'', \text{ since } \cos C \text{ is negative.} \end{aligned}$$

To find b.

A is middle part, $(90^\circ - b)$ and B are adjacent parts.

By Rule 1,

$$\sin A = \cot b \tan B$$

or

$$\cot b = \sin A \cot B$$

$$\begin{aligned} L \sin A &= 9{\cdot}765764 \\ L \cot B &= 10{\cdot}136085 \\ \hline L \cot b &= 9{\cdot}901849 \\ \hline b &= 51^\circ\ 25'\ 15'' \end{aligned}$$

103. (2) Given $A = 120^\circ\ 30'$, $b = 115^\circ\ 35'$, $c = 90^\circ$.

Here we must take account of the signs of the ratios of A and b.

The circular parts are A, B, $(90^\circ - C)$, $(90^\circ - a)$, $(90^\circ - b)$.

To find C.

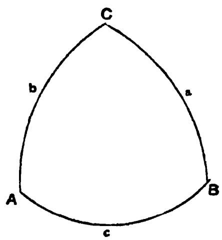
$(90^\circ - b)$ is middle part, A and $(90^\circ - C)$ are adjacent parts.

By Rules 1 and 3,

$$\cos b = -\tan A \cot C$$

or

$$\cot C = \cos b \cot A$$



$$\begin{aligned} L \cos b &= 9{\cdot}635306 \\ L \cot A &= 9{\cdot}770148 \\ \hline L \cot C &= 9{\cdot}405454 \\ \hline & 75^\circ\ 43'\ 45'' \\ & 180 \\ \hline C &= 104^\circ\ 16'\ 15'', \text{ since } \cot C \text{ is negative.} \end{aligned}$$

To find a.

$(90^\circ - a)$ is middle part, A and $(90^\circ - b)$ are opposite parts.

By Rule 2,

$$\begin{aligned}
\overset{-}{\cos a} &= \overset{-}{\cos A}\ \overset{+}{\sin b}\\
L\cos A &= 9{\cdot}705469\\
L\sin b &= 9{\cdot}955186\\
\hline
L\cos a &= 9{\cdot}660655\\
\hline
&\ 62^\circ\ 45'\ 15''\\
&\ 180\\
\hline
a &= 117^\circ\ 14'\ 45'', \text{ since } \cos a \text{ is negative.}
\end{aligned}$$

To find B.

A is middle part, $(90^\circ - b)$ and B are adjacent parts.

By Rule 1,

$$\begin{aligned}
\overset{+}{\sin A} &= \overset{-}{\cot b}\ \overset{-}{\tan B}\\
\text{or}\qquad \tan B &= \sin A \tan b\\
L\sin A &= 9{\cdot}935320\\
L\tan b &= 10{\cdot}319880\\
\hline
L\tan B &= 10{\cdot}255200\\
\hline
&\ 60^\circ\ 56'\ 30''\\
&\ 180\\
\hline
B &= 119^\circ\ \ 3'\ 30'', \text{ since } \tan B \text{ is negative.}
\end{aligned}$$

104. (3) Given $A = 140^\circ\ 20'$, $a = 115^\circ\ 30'$, $c = 90^\circ$.

The triangle is ambiguous (see § 49 (10)), this is evident also from the figure.

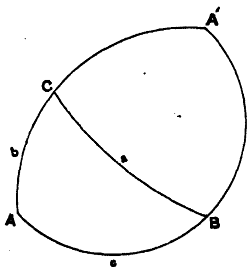
With the exception of A and a the corresponding parts of the two triangles are supplemental.

The circular parts are A, B, $(90^\circ - C)$, $(90^\circ - a)$, $(90^\circ - b)$.

To find B.

B is middle part, $(90^\circ - a)$ and A are adjacent parts.

By Rule 1,



$$\begin{aligned}
\sin B &= \cot a \tan A\\
L\cot a &= 9{\cdot}678496\\
L\tan A &= 9{\cdot}918677\\
\hline
L\sin B &= 9{\cdot}597173\\
\hline
B &= 23^\circ\ 18'\ 0''\\
&\ 180\\
\hline
\text{or} &= 156^\circ\ 42'\ 0''
\end{aligned}$$

To find C.

A is middle part, $(90° - C)$ and $(90° - a)$ are opposite parts.

By Rule 2,

$$\sin A = \sin C \sin a$$

or

$$\sin C = \sin A \operatorname{cosec} a$$

$$\begin{aligned} L \sin A &= 9{\cdot}805038 \\ L \operatorname{cosec} a &= 10{\cdot}044512 \\ \hline L \sin C &= 9{\cdot}849550 \\ \hline C &= 45° \; 0' \; 30'' \\ \hline \text{or} &= 134° \; 59' \; 30'' \end{aligned}$$

To find b.

$(90° - a)$ is middle part, A and $(90° - b)$ are opposite parts.

By Rule 2,

$$\cos a = \cos A \sin b$$

or

$$\sin b = \cos a \sec A$$

$$\begin{aligned} L \cos a &= 9{\cdot}633984 \\ L \sec A &= 10{\cdot}113638 \\ \hline L \sin b &= 9{\cdot}747622 \\ \hline b &= 34° \; 0' \; 15'' \\ & 180 \\ \hline \text{or} &= 145° \; 59' \; 45'' \end{aligned}$$

CHAPTER XI

PRACTICAL SOLUTION OF OBLIQUE-ANGLED SPHERICAL TRIANGLES

105. CASE I. Given the three sides, to find the three angles. Method, using the Tabular Log Haversine Table.

Example.—Given $a = 70^\circ\ 14'\ 20''$, $b = 49^\circ\ 24'\ 10''$, $c = 38^\circ\ 46'\ 10''$, find A, B, C.

First method, using the L Haversine Table.

To find A.

$$\text{hav A} = \text{cosec}\, b\ \text{cosec}\, c \sqrt{\text{hav}\,(a + \overline{b \sim c})\ \text{hav}\,(a - \overline{b \sim c})}$$

$$\text{L hav A} = \text{L cosec}\, b + \text{L cosec}\, c + \tfrac{1}{2}\text{L hav}\,(a + \overline{b \sim c}) + \tfrac{1}{2}\text{L hav}\,(a - \overline{b \sim c}) - 20.$$

$b =$	49°	24′	10″	L cosec	10·119585
$c =$	38	46	10	L cosec	10·203295
$b \sim c =$	10	38	0		
$a =$	70	14	20		
$a + \overline{b \sim c} =$	80	52	20	$\frac{1}{2}$L hav	4·811977
$a - \overline{b \sim c} =$	59	36	20	$\frac{1}{2}$L hav	4·696370
				L hav A	9·831227
				A =	110° 51′ 15″

To find B.

$$\text{hav B} = \text{cosec}\, a\ \text{cosec}\, c \sqrt{\text{hav}\,(b + \overline{a \sim c})\ \text{hav}\,(b - \overline{a \sim c})}$$

$$\text{L hav B} = \text{L cosec}\, a + \text{L cosec}\, c + \tfrac{1}{2}\text{L hav}\,(b + \overline{a \sim c}) + \tfrac{1}{2}\text{L hav}\,(b - \overline{a \sim c}) - 20.$$

$a =$	70°	14′	20″	L cosec	10·026359
$c =$	38	46	10	L cosec	10·203295
$a \sim c =$	31	28	10		
$b =$	49	24	10		
$b + \overline{a \sim c} =$	80	52	20	$\frac{1}{2}$L hav	4·811977
$b - \overline{a \sim c} =$	17	56	0	$\frac{1}{2}$L hav	4·192734
				L hav B	9·234365
				B =	48° 56′ 4″

To find C.

$$\text{hav C} = \text{cosec } a \text{ cosec } b \sqrt{\text{hav }(c + \overline{a \sim b}) \text{ hav }(c - \overline{a \sim b})}$$

$$\text{L hav C} = \text{L cosec } a + \text{L cosec } b + \tfrac{1}{2}\text{L hav }(c + \overline{a \sim b}) + \tfrac{1}{2}\text{L hav }(c - \overline{a \sim b}) - 20.$$

$a = 70° \ 14' \ 20''$	L cosec	10·026359
$b = 49 \ \ 24 \ \ 10$	L cosec	10·119585
$\overline{a \sim b} = 20 \ \ 50 \ \ 10$		
$c = 38 \ \ 46 \ \ 10$		
$c + \overline{a \sim b} = 59 \ \ 36 \ \ 20$	½L hav	4·696370
$c - \overline{a \sim b} = 17 \ \ 56 \ \ 0$	½L hav	4·192734
	L hav C	9·035048
		C = 38° 26′ 47″

106. CASE II. Given two sides and the included angle, to find the third side, and then, by Case I., to find the other two angles.

Example.—Given $a = 68°20'25''$, $b = 52°18'15''$, $C = 117°12'20''$, find c, A, B.

Method, using the Tabular Log Haversine and Tabular versed sine Tables,

$$\text{vers } c = \text{vers }(a \sim b) + \text{vers } \theta \qquad (1),$$

where

$$\text{vers } \theta = \sin a \sin b \text{ vers C}$$

and ∴

$$\text{hav } \theta = \sin a \sin b \text{ hav C}$$

whence

$$\text{L hav } \theta = \text{L sin } a + \text{L sin } b + \text{L hav C} - 20 \qquad (2).$$

$C = 117° \ 12' \ 20''$	L hav 9·862484	Tab vers θ	=	1071608
$a = 68 \ \ 20 \ \ 25$	L sin 9·968199	Tab vers $(a \sim b)$	=	0038912
$b = 52 \ \ 18 \ \ 15$	L sin 9·898324	Tab vers c	=	1110520
		Tab vers 96° 20′	=	313
$a \sim b = 16 \ \ 2 \ \ 10$	L hav θ 9·729007			
		part for 43″		207
	$\theta = 94° \ 6' \ 23''$			$c = 96° \ 20' \ 43''$

Then by Case I.

To find A.

$$\text{hav A} = \text{cosec } b \text{ cosec } c \sqrt{\text{hav }(a + \overline{b \sim c}) \text{ hav }(a - \overline{b \sim c})}$$

$$\text{L hav A} = \text{L cosec } b + \text{L cosec } c + \tfrac{1}{2}\text{L hav }(a + \overline{b \sim c}) + \tfrac{1}{2}\text{L hav }(a - \overline{b \sim c}) - 20.$$

$b=$	$52^\circ\ 18'\ 15''$	L cosec 10·101676
$c=$	$96\ \ 20\ \ 43$	L cosec 10·002669
$c \sim b=$	$44\ \ \ 2\ \ 28$	
$a=$	$68\ \ 20\ \ 25$	
$a+\overline{c \sim b}=$	$112\ \ 22\ \ 53$	$\frac{1}{2}$L hav 4·919545
$a-\overline{c \sim b}=$	$24\ \ 17\ \ 57$	$\frac{1}{2}$L hav 4·323179
		L hav A 9·347069
		A $= 56^\circ\ 16'\ 15''$

Also by Case I.

To find B.

$$\text{hav } B = \text{cosec } a \text{ cosec } c \sqrt{\text{hav }(b+\overline{a\sim c}) \text{ hav }(b-\overline{a\sim c})}$$

$$\text{L hav } B = \text{L cosec } a + \text{L cosec } c + \tfrac{1}{2}\text{L hav }(b+\overline{a\sim c}) + \tfrac{1}{2}\text{L hav }(b-\overline{a\sim c}) - 20.$$

$a=$	$68^\circ\ 20'\ 25''$	L cosec 10·031801
$c=$	$96\ \ 20\ \ 43$	L cosec 10·002669
$c \sim a=$	$28\ \ \ 0\ \ 18$	
$b=$	$52\ \ 18\ \ 15$	
$b+\overline{c \sim a}=$	$80\ \ 18\ \ 33$	$\frac{1}{2}$L hav 4·809460
$b-\overline{c \sim a}=$	$24\ \ 17\ \ 57$	$\frac{1}{2}$L hav 4·323179
		L hav B 9·167109
		B $= 45^\circ\ 4'\ 41''$

107. Case III. Given the three angles to find the three sides.

(3) Given $A = 109^\circ\ 45'\ 40''$, $B = 130^\circ\ 35'\ 50''$, $C = 141^\circ\ 13'\ 50''$, find a, b, c.

The supplements of the three angles will be the corresponding sides of the polar triangle A′B′C′.

$A = 109^\circ\ 45'\ 40''$	$B = 130^\circ\ 35'\ 50''$	$C = 141^\circ\ 13'\ 50''$
180	180	180
$a' = 70^\circ\ 14'\ 20''$	$b' = 49^\circ\ 24'\ 10''$	$c' = 38^\circ\ 46'\ 10''$

With the three sides a', b', c', find the three angles A′, B′, C′ by the method of Case I. The supplements of these angles will be the corresponding sides of the primitive triangle ABC.

Thus we find

$A' = 110^\circ\ 51'\ 15''$	$B' = 48^\circ\ 56'\ 4''$	$C' = 38^\circ\ 26'\ 47''$
180	180	180
$a = 69^\circ\ 8'\ 45''$	$b = 131^\circ\ 3'\ 56''$	$c = 141^\circ\ 33'\ 13''$

108. Case IV. Given two angles and the interjacent side, to find the third angle.

(4) Given $A = 111^\circ\ 39'\ 35''$, $B = 127^\circ\ 41'\ 45''$, $c = 62^\circ\ 47'\ 40''$, find a, b, C.

The supplements of these two angles and interjacent side will be the two corresponding sides and included angle of the polar triangle A′B′C′.

$A = 111^\circ\ 39'\ 35''$	$B = 127^\circ\ 41'\ 45''$	$c = 62^\circ\ 47'\ 40''$
180	180	180
$a' = 68^\circ\ 20'\ 25''$	$b' = 52^\circ\ 18'\ 15''$	$C' = 117^\circ\ 12'\ 20''$

By the method of Case II., find c',
and, by the method of Case I., find A′, B′.

The supplements of these parts will be the two sides and remaining angle of the primitive triangle ABC,

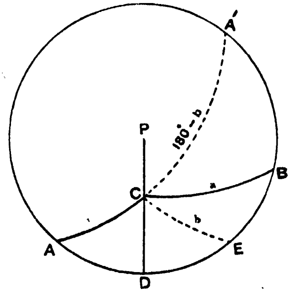
thus $c' = 96^\circ\ 20'\ 43''$	$A' = 56^\circ\ 16'\ 15''$	$B' = 45^\circ\ 4'\ 41''$
180	180	180
$C = 83^\circ\ 39'\ 17''$	$a = 123^\circ\ 43'\ 45''$	$b = 134^\circ\ 55'\ 19''$

109. Case V. (1) Given $A = 42^\circ\ 35'\ 30''$, $b = 61^\circ\ 25'\ 45''$, $a = 70^\circ\ 30'\ 15''$.

Here a lies between b and $180^\circ - b$. Hence the triangle is unique, and B will be of *like* affection with b (see § 46).

$$\sin B = \frac{\sin b}{\sin a} \sin A = \sin b \operatorname{cosec} a \sin A.$$

L sin b	9·943607
L cosec a	10·025642
L sin A	9·830440
L sin B	9·799689
	$B = 39^\circ\ 5'\ 15''$



From C draw CD perpendicular to AB; it will be of like affection with A, and will fall inside the triangle ACB (§ 49 (3)).

In the triangle ACD.

$$\cos A = \cot b \tan AD \qquad\qquad \cos b = \cot A \cot ACD$$

$$\tan AD = \cos A \tan b \qquad\qquad \cot ACD = \cos b \tan A$$

L cos A	9·866993	L cos b	9·679650
L tan b	10·263956	L tan A	9·963447
L tan AD	10·130949	L cot ACD	9·643097
	$AD = 53^\circ\ 30'\ 30''$		$ACD = 66^\circ\ 16'\ 0''$

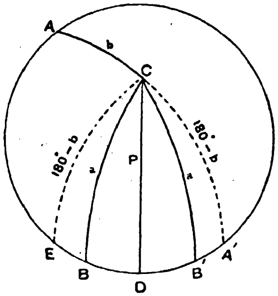
In the triangle BCD.

$\cos B = \cot a \tan BD$		$\cos a =$	$\cot B \cot BCD$
$\tan BD = \cos B \tan a$		$\cot BCD =$	$\cos a \tan B$
L cos B	9·889965	L cos a	9·523406
L tan a	10·450952	L tan B	9·909725
L tan BD	10·340917	L cot BCD	9·433131
BD =	65° 28′ 45″	BCD =	74° 50′ 0″
AD =	53 30 30	ACD =	66 16 0

$c = AD + BD = 118° 59' 15''$ $\quad C = ACD + BCD = 141° 6' 0''$

110. CASE V. (2) Given $A = 137° 24' 30''$, $b = 61° 25' 45''$, $a = 125° 34' 15''$.

Here a does not lie between b and $180° - b$, hence the triangle is ambiguous, and the two values of B will be supplemental (see § 46).



$$\sin B = \frac{\sin b}{\sin a} \sin A = \sin b \operatorname{cosec} a \sin A.$$

L sin b =	9·943607
L cosec a =	10·089697
L sin A =	9·830440
L sin B =	9·863744
B =	46° 56′ 45″
B′ =	133 3 15

From C draw CD perpendicular to AB. It will be of like affection with A and will bisect the angle BCB′ between the two positions of the side a (§ 49 (12)).

In the triangle ACD.

$$\overset{-}{\cos A} = \overset{+}{\cot b}\ \overset{-}{\tan AD} \qquad \overset{+}{\cos b} = \overset{-}{\cot A}\ \overset{-}{\cot ACD}$$

$\tan AD = \cos A \tan b$		$\cot ACD = \cos b \tan A$	
L cos A	9·866993	L cos b	9·679650
L tan b	10·263956	L tan A	9·963447
L tan AD =	10·130949	L cot ACD =	9·643097
	53° 30′ 30″		66° 16′ 0″
	180		180
AD =	126° 29′ 30″	ACD =	113° 44′ 0″

since tan AD is negative. $\qquad$ since cot ACD is negative.

In the triangle B'CD.

$$\overset{-}{\cos a} = \overset{+}{\cot B'CD}\ \overset{-}{\cot CB'D} \qquad \overset{-}{\cos CB'D} = \overset{-}{\cot a}\ \overset{+}{\tan B'D}$$

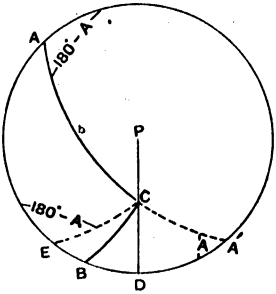
$$\begin{array}{rl} \cot B'CD = \cos a \tan CB'D & \\ L \cos a & 9{\cdot}764706 \\ L \tan CB'D & 10{\cdot}029521 \\ \hline L \cot B'CD & 9{\cdot}794227 \\ \hline BCD = B'CD & 58^\circ\ 5'\ 30'' \\ ACD = & 113\ \ 44\ \ \ 0 \\ \hline C = ACD + B'CD = & 171\ \ 49\ \ 30 \\ \text{or} = ACD - BCD = & 55\ \ 38\ \ 30 \\ \hline \end{array} \qquad \begin{array}{rl} \tan B'D = \cos CB'D \tan a & \\ L \cos CB'D & 9{\cdot}834223 \\ L \tan a & 10{\cdot}145597 \\ \hline L \tan B'D & 9{\cdot}979820 \\ \hline BD = B'D & 43^\circ\ 40'\ 15'' \\ AD = & 126\ \ 29\ \ 30 \\ \hline c = AD + B'D = & 170\ \ \ 9\ \ 45 \\ \text{or} = AD - BD = & 82\ \ 49\ \ 15 \\ \hline \end{array}$$

111. Case VI. (1) Given $A = 42^\circ\ 35'\ 30''$, $b = 118^\circ\ 34'\ 15''$, $B = 130^\circ\ 16'\ 45''$.

Here B lies between A and $180^\circ - A$. Hence the triangle is unique, and a will be of *like* affection with A (see § 46).

$$\sin a = \frac{\sin A}{\sin B} \sin b = \sin A \operatorname{cosec} B \sin b.$$

$$\begin{array}{ll} L \sin A & 9{\cdot}830440 \\ L \operatorname{cosec} B & 10{\cdot}117530 \\ L \sin b & 9{\cdot}943607 \\ \hline L \sin a & 9{\cdot}891577 \\ \hline & a = 51^\circ\ 10'\ 30'' \\ \hline \end{array}$$



From C draw CD perpendicular to AB; it will be of like affection with A and will fall outside the triangle ACB (§ 49 (4)).

In the triangle ACD.

$$\overset{+}{\cos A} = \overset{-}{\cot b}\ \overset{-}{\tan AD} \qquad \overset{-}{\cos b} = \overset{+}{\cot A}\ \overset{-}{\cot ACD}$$

$$\begin{array}{rl} \tan AD = \cos A \tan b & \\ L \cos A & 9{\cdot}866993 \\ L \tan b & 10{\cdot}263956 \\ \hline L \tan AD & 10{\cdot}130949 \\ \hline & 53^\circ\ 30'\ 30'' \\ & 180 \\ \hline AD = & 126^\circ\ 29'\ 30'' \end{array} \qquad \begin{array}{rl} \cot ACD = \cos b \tan A & \\ L \cos b & 9{\cdot}679650 \\ L \tan A & 9{\cdot}963447 \\ \hline L \cot ACD & 9{\cdot}643097 \\ \hline & 66^\circ\ 16'\ 0'' \\ & 180 \\ \hline ACD = & 113^\circ\ 44'\ 0'' \end{array}$$

since tan AD is negative. since cot ACD is negative.

In the triangle BCD.

$\overset{+}{\cos}\,\mathrm{CBD} = \overset{+}{\cot}\,a\ \overset{+}{\tan}\,\mathrm{BD}$

$\tan \mathrm{BD} = \cos \mathrm{CBD} \tan a$

L cos CBD	9·810577
L tan a	10·094345
L tan BD	9·904922
BD =	38° 46′ 45″
AD =	126 29 30
c = AD − BD =	87° 42′ 45″

$\overset{+}{\cos}\,a = \overset{+}{\cot}\,\mathrm{CBD}\ \overset{+}{\cot}\,\mathrm{BCD}$

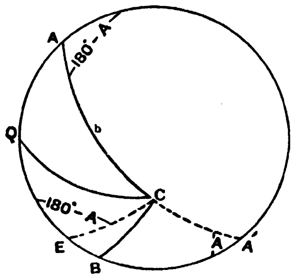
$\cot \mathrm{BCD} = \cos a \tan \mathrm{CBD}$

L cos a =	9·797229
L tan CBD =	10·071893
L cot BCD =	9·869122
BCD =	53° 30′ 15″
ACD =	113 44 0
C = ACD − BCD =	60° 13′ 45″

112. Case VI. Another method.

(1) Given A = 42° 35′ 30″, b = 118° 34′ 15″, B = 130° 16′ 45″, as before a = 51° 10′ 30″ (see §§ 46, 49 (5)).

From C draw to AB the quadrant CQ. It will fall inside the triangle ACB since a and b are of unlike affection (§ 49 (6)).



In the triangle ACQ.

$\overset{+}{\cos}\,\mathrm{A} = -\overset{-}{\cot}\,b\ \overset{+}{\cot}\,\mathrm{AQ}$

$\cot \mathrm{AQ} = \cos \mathrm{A} \tan b$

L cos A =	9·866993
L tan b =	10·263956
L cot AQ =	10·130949
AQ =	36° 29′ 30″

$\overset{-}{\cos}\,b = -\overset{+}{\cot}\,\mathrm{A}\ \overset{+}{\tan}\,\mathrm{ACQ}$

$\tan \mathrm{ACQ} = \cos b \tan \mathrm{A}$

L cos b	9·679650
L tan A	9·963447
L tan ACQ =	9·643097
ACQ =	23° 44′ 0″

In the triangle BCQ.

$$\begin{aligned} \overset{-}{\cos \mathrm{B}} &= -\overset{+}{\cot a}\,\overset{+}{\cot \mathrm{BQ}} \\ \cot \mathrm{BQ} &= \cos \mathrm{B} \tan a \\ \mathrm{L} \cos \mathrm{B} &= 9{\cdot}810572 \\ \mathrm{L} \tan a &= 10{\cdot}094345 \\ \hline \mathrm{L} \cot \mathrm{BQ} &= 9{\cdot}904917 \\ \hline \mathrm{BQ} &= 51^\circ\ 13'\ 15'' \\ \mathrm{AQ} &= 36\ \ 29\ \ 30 \\ \hline c = \mathrm{AQ} + \mathrm{BQ} &= 87^\circ\ 42'\ 45'' \end{aligned}$$

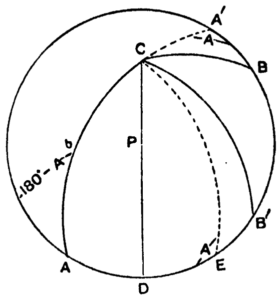
$$\begin{aligned} \overset{+}{\cos a} &= -\overset{-}{\cot \mathrm{B}}\,\overset{+}{\tan \mathrm{BCQ}} \\ \tan \mathrm{BCQ} &= \cos a \tan \mathrm{B} \\ \mathrm{L} \cos a &= 9{\cdot}797229 \\ \mathrm{L} \tan \mathrm{B} &= 10{\cdot}071893 \\ \hline \mathrm{L} \tan \mathrm{BCQ} &= 9{\cdot}869122 \\ \hline \mathrm{BCQ} &= 36^\circ\ 29'\ 45'' \\ \mathrm{ACQ} &= 23\ \ 44\ \ \ 0 \\ \hline \mathrm{C} = \mathrm{ACQ} + \mathrm{BCQ} &= 60^\circ\ 13'\ 45'' \end{aligned}$$

113. CASE VI. (2) Given $\mathrm{A} = 137^\circ\ 24'\ 30''$, $b = 118^\circ\ 34'\ 15''$, $\mathrm{B} = 140^\circ\ 10'$.

Here B does not lie between A and $180^\circ - \mathrm{A}$. Hence the triangle is ambiguous, and the two values of a will be supplemental (see § 46).

$$\sin a = \frac{\sin \mathrm{A}}{\sin \mathrm{B}} \sin b = \sin \mathrm{A} \operatorname{cosec} \mathrm{B} \sin b.$$

$$\begin{aligned} \mathrm{L} \sin \mathrm{A} &= 9{\cdot}830440 \\ \mathrm{L} \operatorname{cosec} \mathrm{B} &= 10{\cdot}193443 \\ \mathrm{L} \sin b &= 9{\cdot}943607 \\ \hline \mathrm{L} \sin a &= 9{\cdot}967490 \\ \hline a &= 68^\circ\ \ 6'\ 15'' \\ & \ \ 180 \\ \hline \text{or} &= 111^\circ\ 53'\ 45'' \end{aligned}$$



From C draw CD perpendicular to AB, it will be of *like* affection with A; also the angles DCB, DCB′ and the sides DB, DB′ will be supplemental (§ 45 (5)).

In the triangle ACD.

$$\begin{aligned} \overset{-}{\cos \mathrm{A}} &= \overset{-}{\cot b}\,\overset{+}{\tan \mathrm{AD}} \\ \tan \mathrm{AD} &= \cos \mathrm{A} \tan b \\ \mathrm{L} \cos \mathrm{A} &\quad 9{\cdot}866993 \\ \mathrm{L} \tan b &\quad 10{\cdot}263956 \\ \hline \mathrm{L} \tan \mathrm{AD} &\quad 10{\cdot}130949 \\ \hline \mathrm{AD} &= 53^\circ\ 30'\ 30'' \end{aligned}$$

$$\begin{aligned} \overset{-}{\cos b} &= \overset{-}{\cot \mathrm{A}}\,\overset{+}{\cot \mathrm{ACD}} \\ \cot \mathrm{ACD} &= \cos b \tan \mathrm{A} \\ \mathrm{L} \cos b &\quad 9{\cdot}679650 \\ \mathrm{L} \tan \mathrm{A} &\quad 9{\cdot}963447 \\ \hline \mathrm{L} \cot \mathrm{ACD} &\quad 9{\cdot}643097 \\ \hline \mathrm{ACD} &= 66^\circ\ 16'\ 0'' \end{aligned}$$

In the triangle BCD.

$\overset{-}{\cos \mathrm{CBD}} = \overset{+}{\cot a}\ \overset{-}{\tan \mathrm{BD}}$

$\tan \mathrm{BD} = \cos \mathrm{CBD} \tan a$

L cos CBD	9·885311
L tan a	10·395868
L tan BD	10·281179
B′D =	62° 22′ 15″
	180
BD =	117° 37′ 45″

since tan BD is neg.

AD =	53 30 30
c = AD + BD =	171° 8′ 15″
or = AD + B′D =	115 52 45

$\overset{+}{\cos a} = \overset{-}{\cot \mathrm{BCD}}\ \overset{-}{\cot \mathrm{CBD}}$

$\cot \mathrm{BCD} = \cos a \tan \mathrm{CBD}$

L cos a	9·571616
L tan CBD	9·921247
L cot BCD	9·492863
B′CD	72° 43′ 15″
	180
BCD	107° 16′ 45″

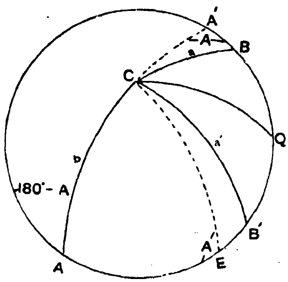
since cot BCD is neg.

ACD =	66 16 0
C′ = ACD + BCD =	173° 32′ 45″
or = ACD + B′CD =	138 59 15

114. Case VI. Another method.

(2) Given A = 137° 24′ 30″, b = 118° 34′ 15″, B = 140° 10′.

As before a = 68° 6′ 15″, or = 111° 53′ 45″.



From C draw to AB the quadrant CQ, it will bisect the angle BCB′ and side BB′ (see §§ 46, 49 (13)).

In the triangle ACQ.

$\overset{-}{\cos \mathrm{A}} = -\overset{-}{\cot b}\ \overset{-}{\cot \mathrm{AQ}}$

$\cot \mathrm{AQ} = \cos \mathrm{A} \tan b$

L cos A	9·866993
L tan b	10·263956
L cot AQ	10·130949
	36° 29′ 30″
	180
AQ =	143° 30′ 30″

since cot AQ is negative.

$\overset{-}{\cos b} = -\overset{-}{\cot \mathrm{A}}\ \overset{-}{\tan \mathrm{ACQ}}$

$\tan \mathrm{ACQ} = \cos b \tan \mathrm{A}$

L cos b	9·679650
L tan A	9·963447
L tan ACQ	9·643097
	23° 44′ 0″
	180
ACQ =	156° 16′ 0″

since tan ACQ is negative.

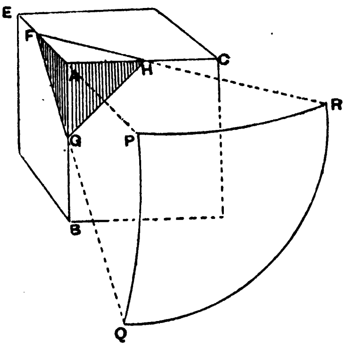
In the triangle BCQ.

$$\begin{array}{rl} \cos B = & -\cot BQ \cot a \\ \cot BQ = & \cos B \tan a \\ L \cos B = & 9{\cdot}885311 \\ L \tan A = & 10{\cdot}395868 \\ \hline L \cot BQ = & 10{\cdot}281179 \\ \hline BQ = B'Q = & 27^\circ\ 37'\ 45'' \\ AQ = & 143\ \ 30\ \ 30 \\ \hline c = AQ + BQ = & 171^\circ\ \ 8'\ 15'' \\ \text{or} = AQ - B'Q = & 115\ \ 52\ \ 45 \end{array} \qquad \begin{array}{rl} \cos a = & -\tan BCQ \cot B \\ \tan BCQ = & \cos a \tan B \\ L \cos a = & 9{\cdot}571616 \\ L \tan B = & 9{\cdot}921247 \\ \hline L \tan BCQ = & 9{\cdot}492863 \\ \hline BCQ = B'CQ = & 17^\circ 16' 45'' \\ ACQ = & 156\ \ 16\ \ 0 \\ \hline C = ACQ + BCQ = & 173^\circ\ 32'\ 45'' \\ \text{or} = ACQ - B'CQ = & 138\ \ 59\ \ 15 \end{array}$$

115. The following examples may be useful as illustrating the method of dealing with the inclinations of planes.

Example 1. Through three points taken on the edges of a cube equidistant from one angle of the cube a section is made. Find the angle between the cutting plane and a side of the cube.

In the figure AG, AH, AF are all equal and at right angles, therefore FAH, FAG, GAH are isosceles right-angled triangles, also FG, GH, HF being all equal FHG is equilateral.



Imagine F to be centre of a sphere, and suppose the planes AFH, AFG, GFH to cut the surface of this sphere in the arcs of great circles PR, PQ, RQ forming the spherical triangle PRQ, in which

$$PR = AFH = 45^\circ,$$
$$PQ = AFG = 45^\circ,$$
$$RQ = GFH = 60^\circ.$$

To find the angle between the plane GFH and the side of the cube AFH we have this angle = PRQ.

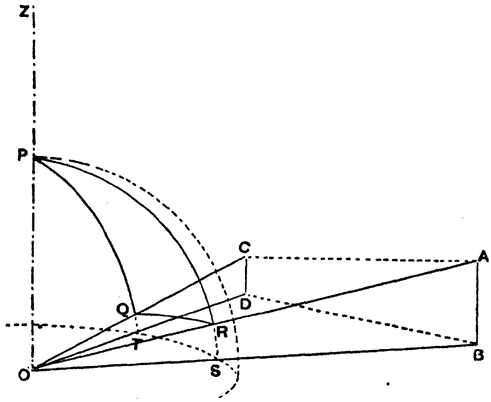
$$\cos PRQ = \frac{\cos 45^\circ - \cos 60^\circ \cos 45^\circ}{\sin 60^\circ \sin 45^\circ} = \frac{\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2}}{\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{\sqrt{6}} = \frac{1}{\sqrt{3}}.$$

$$\begin{aligned} L \cos PRQ &= 10 - \tfrac{1}{2}\log 3 \\ &= 10 - {\cdot}238560 \\ &= 9{\cdot}761440 \end{aligned}$$

$$\therefore PRQ = 54^\circ\ 44'$$

Example 2. The elevation of a spire is 20° and of a tower is 12°, and the angular distance between their summits is 50°; find the horizontal angle between their positions.

In the figure O is the position of the observer.



AB is the vertical height of the spire above the horizontal plane,
CD is the vertical height of the tower above the horizontal plane,
AOB = elevation of the spire = 20°,
COD = elevation of the tower = 12°,
AOC = angular distance of their summits = 50°,
OZ is the vertical line through the observer's position, meeting the celestial concave in the zenith Z.

Hence ZOD, ZOB are right angles and therefore BOD is the horizontal angle between the vertical planes AOB, COD.

Imagine a sphere with O as centre and of any radius. The planes ZOB, ZOD, COA will cut the surface of this sphere in three arcs of great circles forming a spherical triangle PQR, of which

$$\begin{aligned} PQ &= 90^\circ - QT = 90^\circ - COD = 90^\circ - 12^\circ = 78^\circ, \\ PR &= 90^\circ - RS = 90^\circ - AOB = 90^\circ - 20^\circ = 70^\circ, \\ RQ &= COA \qquad\quad\; = 50^\circ. \end{aligned}$$

The horizontal angle BOD being the angle between the two planes ZOB, ZOD = the spherical angle RPQ, and we have

78°	L cosec	10·009596
70°	L cosec	10·027014
8		
50		
58	$\frac{1}{2}$ L hav	4·685571
42	$\frac{1}{2}$ L hav	4·554329
	L hav RPQ	9·276510

RPQ = 51° 32′ 30″ = horizontal angle BOD.

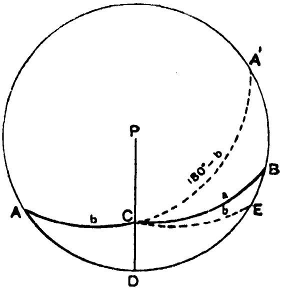
116. In order to further illustrate Cases V. and VI. of the solution of oblique-angled spherical triangles, the following sketch solutions are added.

In all cases the figure is described on the plane of that side of the spherical triangle which does not enter into the Rule of Sines. Thus

Given A, C, a, the figure is described on the plane of AC (or b).
Given a, b, B, " " " AB (or c).

The perpendicular from the third angle to the opposite side is always found by joining the pole of that side—on the plane of which the figure is described—with the extremity of the given side adjacent to the given angle, and producing this arc to meet the opposite side (see § 46).

(1) $A=30°$, $b=65°$, $a=80°$.

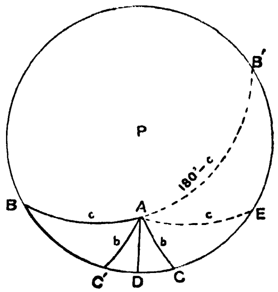


Here a lies between b and $180° - b$, hence there is only one triangle having the given parts, and B will be of like affection with b.

Also the perpendicular CD will fall inside the triangle ABC, since A and B are of like affection,

$$\therefore \quad C = ACD + BCD,$$
$$c = AD + BD.$$

(2) $B=30°$, $c=65°$, $b=40°$.

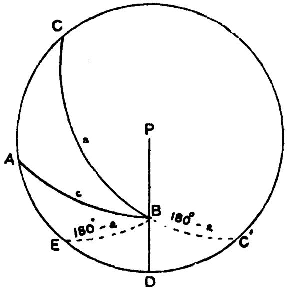


Here b does not lie between c and $180° - c$, hence there are two triangles having the given parts, and the two values of C will be supplemental.

Also the perpendicular AD will fall midway between the two positions of the side b,

$$\therefore \quad A = BAD \pm CAD,$$
$$a = BD \pm CD.$$

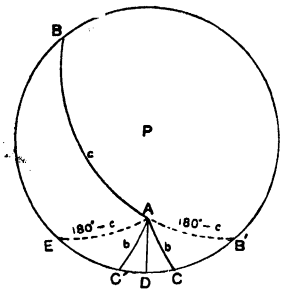
(3) $C=40°$, $a=120°$, $c=70°$.



Here c lies between a and $180° - a$, hence there is only one triangle having the given parts, and A will be of like affection with a.

Also the perpendicular BD will fall outside the triangle ABC, since C and A are of unlike affection,

$$\therefore \quad B = CBD - ABD,$$
$$b = CD - AD.$$

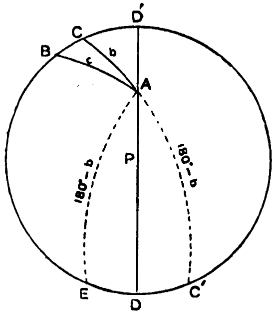


(4) $B = 40°$, $c = 120°$, $b = 50°$.

Here b does not lie between c and $180° - c$,
hence there are two triangles having the given parts, and the two values of C will be supplemental.

Also the perpendicular will fall midway between the two positions of the side b,

$\therefore$ $$A = BAD \pm CAD,$$
$$a = BD \pm CD.$$

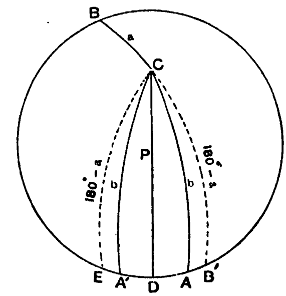


(5) $C = 120°$, $c = 70°$, $b = 60°$.

Here c lies between b and $180° - b$,
hence there is only one triangle having the given parts, and B will be of like affection with b.

Also the perpendicular AD′ will fall outside the triangle ABC opposite the acute angle B,

$\therefore$ $$A = BAD' - CAD',$$
$$a = BD' - CD'.$$

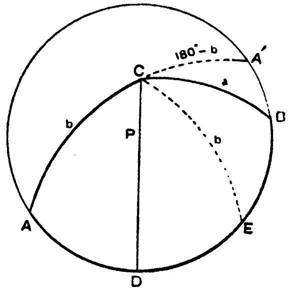


(6) $B = 120°$, $a = 60°$, $b = 125°$.

Here b does not lie between a and $180° - a$,
hence there are two triangles having the given parts, and the two values of A will be supplemental.

Also the perpendicular CD will fall midway between the two positions of the side b,

$\therefore$ $$C = BCD \pm ACD,$$
$$c = BD \pm AD.$$



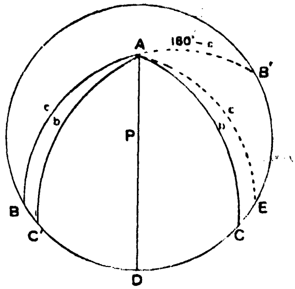
(7) $A = 130°$, $b = 110°$, $a = 80°$.

Here a lies between b and $180° - b$,
hence there is only one triangle having the given parts, and B will be of like affection with b.

Also the perpendicular CD will fall inside the triangle ABC, since A and B are of like affection,

$\therefore$ $$C = ACD + BCD,$$
$$c = AD + BD.$$

(8) $B = 150°$, $c = 115°$, $b = 120°$.



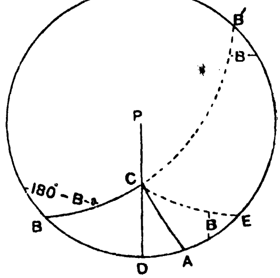
Here b does not lie between c and $180° - c$,
hence there are two triangles having the given parts, and the two values of C will be supplemental.

Also the perpendicular AD will fall midway between the two positions of the side b,

$\therefore \quad A = BAD \pm CAD,$

$\quad a = BD \pm CD.$

(9) $A = 70°$, $B = 50°$, $a = 60°$.



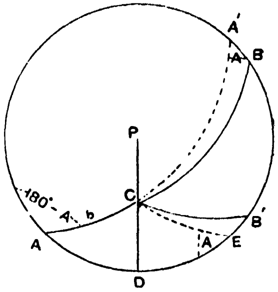
A lies between B and $180° - B$, hence there is only one triangle having the given parts, and b will be of like affection with B.

Also the perpendicular CD falls inside the triangle ABC, since A and B are of like affection,

$\therefore \quad C = ACD + BCD,$

$\quad c = AD + BD.$

(10) $A = 50°$, $B = 45°$, $b = 60°$.



B does not lie between A and $180° - A$, hence there are two triangles having the given parts, and the two values of a are supplemental.

Also the perpendicular CD falls inside the triangles ACB, ACB′, because A and B (or B′) are of like affection,

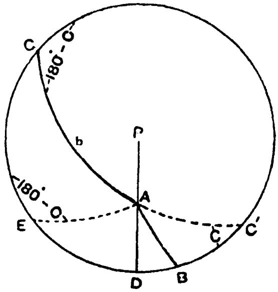
$\therefore \quad C = ACD + DCB$

or $= ACD + DCB' = ACD + (180° - DCB)$

$\quad c = AD + DB$

or $\quad = AD + DB' = AD + (180° - DB).$

(11) $C = 45°$, $b = 110°$, $B = 50°$.

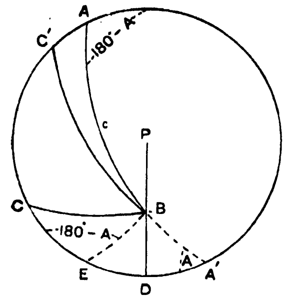


B lies between C and $180° - C$, hence there is only one triangle having the given parts, and c will be of like affection with C.

Also the perpendicular AD falls inside the triangle ABC, since B and C are of like affection,

$\therefore \quad A = CAD + BAD,$

$\quad a = CD + BD.$

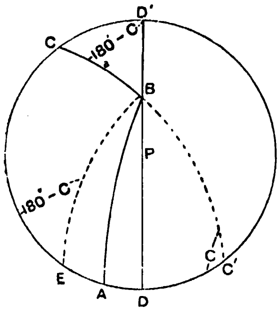


(12) $A = 60°$, $C = 124°$, $c = 110°$.

C does not lie between A and $180° - A$, hence there are two triangles having the given parts, and the two values of a will be supplemental.

Also the perpendicular BD falls outside the triangles ABC, ABC′ opposite the acute angle A,

$$\therefore \quad B = ABD - CBD$$
$$\text{or} = ABD - C'BD = ABD - (180° - CBD)$$
$$b = AD - CD$$
$$\text{or} \quad = AD - C'D = AD - (180° - CD).$$

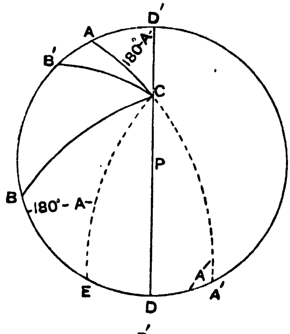


(13) $A = 75°$, $C = 120°$, $a = 50°$.

A lies between C and $180° - C$, hence there is only one triangle having the given parts, and c will be of like affection with C.

Also the perpendicular BD′ will fall outside the triangle ABC opposite the acute angle A,

$$\therefore B = ABD' - CBD', \; b = AD' - CD'.$$

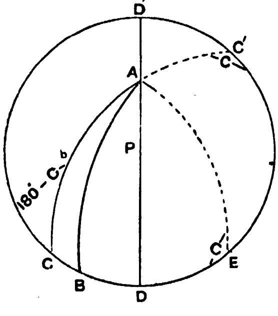


(14) $B = 45°$, $A = 120°$, $b = 50°$.

B does not lie between A and $180° - A$, hence there are two triangles having the given parts, and the two values of a will be supplemental.

Also the perpendicular CD′ will fall outside the triangles ABC, AB′C, opposite the acute angle B (or B′),

$$\therefore \quad C = BCD' - ACD'$$
$$\text{or} \quad = B'CD' - ACD' = 180° - BCD' - ACD',$$
$$c = BD' - AD'$$
$$\text{or} \quad = B'D' - AD' = 180° - BD' - AD'.$$



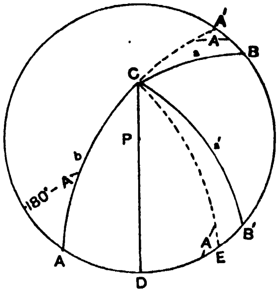
(15) $C = 135°$, $b = 120°$, $B = 60°$.

B lies between C and $180° - C$, hence there is only one triangle having the given parts, and c will be of like affection with C.

Also the perpendicular AD′ falls outside the triangle ABC opposite the acute angle B,

$$\therefore A = BAD' - CAD', \; a = BD' - CD'.$$

(16) $A = 135°$, $b = 125°$, $B = 140°$.



B does not lie between A and $180° - A$, hence there are two triangles having the given parts, and the two values of a will be supplemental.

Also the perpendicular CD falls inside the triangles ABC, AB′C, since A and B (or B′) are of like affection,

$$\therefore \qquad C = ACD + BCD$$

$$\text{or} = ACD + B'CD = ACD + (180° - BCD),$$

$$c = AD + BD$$

$$\text{or} \quad = AD + B'D = AD + (180° - BD).$$

CHAPTER XII

SOLUTION OF OBLIQUE-ANGLED SPHERICAL TRIANGLES BY METHODS NOT REQUIRING THE L HAVERSINE OR TABULAR VERSED SINE TABLES OR NAPIER'S RULES

117. CASE I. **First method.**

Given $a = 70° \ 14' \ 20''$, $b = 49° \ 24' \ 10''$, $c = 38° \ 46' \ 10''$, find A, B, C.

To find A.

$$\sin\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}} = \sqrt{\sin(s-b)\sin(s-c)\operatorname{cosec} b \operatorname{cosec} c}$$

$$L\sin\frac{A}{2} = \tfrac{1}{2}\{L\sin(s-b) + L\sin(s-c) + L\operatorname{cosec} b + L\operatorname{cosec} c\} - 10.$$

$$\begin{array}{rrrrl}
a = 70° & 14' & 20'' & & s-a = \ \ 8° \ 58' \ \ 0'' \\
b = 49 & 24 & 10 & & s-b = 29 \ \ 48 \ \ 10 \\
c = 38 & 46 & 10 & & s-c = 40 \ \ 26 \ \ 10 \\
\hline
2)158 & 24 & 40 & & \\
\hline
s = 79° & 12' & 20'' & & \\
\hline
\end{array}$$

$$\begin{array}{rr}
L\sin(s-b) & 9{\cdot}696370 \\
L\sin(s-c) & 9{\cdot}811977 \\
L\operatorname{cosec} b & 10{\cdot}119585 \\
L\operatorname{cosec} c & 10{\cdot}203295 \\
\hline
& 2)39{\cdot}831227 \\
\hline
L\sin\frac{A}{2} & 9{\cdot}915613 \\
\hline
\frac{A}{2} & 55° \ 25' \ 37''{\cdot}5 \\
\hline
A & 110° \ 51' \ 15''
\end{array}$$

To find B.

$$\sin\frac{B}{2}=\sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin a\sin c}}=\sqrt{\sin(s-a)\sin(s-c)\operatorname{cosec}a\operatorname{cosec}c},$$

$$L\sin\frac{B}{2}=\tfrac{1}{2}\{L\sin(s-a)+L\sin(s-c)+L\operatorname{cosec}a+L\operatorname{cosec}c\}-10.$$

$L\sin(s-a)$	9·192734
$L\sin(s-c)$	9·811977
$L\operatorname{cosec}a$	10·026359
$L\operatorname{cosec}c$	10·203295
	2)39·234365
$L\sin\frac{B}{2}$	9·617183
$\frac{B}{2}$	24° 28′ 2″·4
B	48° 56′ 5″

To find C.

$$\sin\frac{C}{2}=\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}}=\sqrt{\sin(s-a)\sin(s-b)\operatorname{cosec}a\operatorname{cosec}b},$$

$$L\sin\frac{C}{2}=\tfrac{1}{2}\{L\sin(s-a)+L\sin(s-b)+L\operatorname{cosec}a+L\operatorname{cosec}b\}-10.$$

$L\sin(s-a)$	9·192734
$L\sin(s-b)$	9·696370
$L\operatorname{cosec}a$	10·026359
$L\operatorname{cosec}b$	10·119585
	2)39·035048
$L\sin\frac{C}{2}$	9·517524
$\frac{C}{2}$	19° 13′ 23″·4
C	38° 26′ 47″

118. Case I. Second method—the same example.

To find A.

$$\tan\frac{A}{2}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}},$$

$$L\tan\frac{A}{2}=10+\tfrac{1}{2}\{L\sin(s-b)+L\sin(s-c)-L\sin s-L\sin(s-a)\}.$$

$L\sin(s-b)$	9·696370	$L\sin s$	9·992247
$L\sin(s-c)$	9·811977	$L\sin(s-a)$	9·192734
			19·184981

	19·508347
	19·184982
	2) ·323366
$L\tan\frac{A}{2}$	10·161683
$\frac{A}{2}$ =	55° 25′ 37″·7
A =	110° 51′ 15″

To find B.

$$\tan\frac{B}{2}=\sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin s\sin(s-b)}},$$

$$L\tan\frac{B}{2}=10+\tfrac{1}{2}\{L\sin(s-a)+L\sin(s-c)-L\sin s-L\sin(s-b)\}.$$

$$\begin{array}{lr}
L\sin(s-a) & 9{\cdot}192734 \\
L\sin(s-c) & 9{\cdot}811977 \\
\hline
 & 19{\cdot}004711 \\
 & 19{\cdot}688617 \\
\hline
 & 2)\overline{1}{\cdot}316094 \\
\hline
L\tan\frac{B}{2}= & 9{\cdot}658047 \\
\hline
\frac{B}{2}= & 24^\circ\ 28'\ 2'' \\
\hline
B= & 48^\circ\ 56'\ 4''
\end{array}
\qquad
\begin{array}{lr}
L\sin s & 9{\cdot}992247 \\
L\sin(s-b) & 9{\cdot}696370 \\
\hline
 & 19{\cdot}688617
\end{array}$$

To find C.

$$\tan\frac{C}{2}=\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s\sin(s-c)}},$$

$$L\tan\frac{C}{2}=10+\tfrac{1}{2}\{L\sin(s-a)+L\sin(s-b)-L\sin s-L\sin(s-c)\}.$$

$$\begin{array}{lr}
L\sin(s-a) & 9{\cdot}192734 \\
L\sin(s-b) & 9{\cdot}696370 \\
\hline
 & 18{\cdot}889104 \\
 & 19{\cdot}804224 \\
\hline
 & 2)\ \overline{1}{\cdot}084880 \\
\hline
L\tan\frac{C}{2}= & 9{\cdot}542440 \\
\hline
\frac{C}{2}= & 19^\circ\ 13'\ 23''{\cdot}4 \\
\hline
C= & 38^\circ\ 26'\ 47''
\end{array}
\qquad
\begin{array}{lr}
L\sin s & 9{\cdot}992247 \\
L\sin(s-c) & 9{\cdot}811977 \\
\hline
 & 19{\cdot}804224
\end{array}$$

119. Case I. Another form of the second method is as follows —the same example.

To find A.

$$\tan\frac{A}{2}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}=\frac{1}{\sin(s-a)}\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}$$

$$=\frac{X}{\sin(s-a)},$$

where
$$X=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}},$$

$$L\tan\frac{A}{2}=10+\tfrac{1}{2}\{L\sin(s-a)+L\sin(s-b)+L\sin(s-c)-L\sin s\}-L\sin(s-a)$$
$$=10+\log X-L\sin(s-a).$$

$L\sin(s-a)$	9·192734
$L\sin(s-b)$	9·696370
$L\sin(s-c)$	9·811977
	28·701081
$L\sin s$	9·992247
	2)18·708834
$10+\log X$	19·354417
$L\sin(s-a)$	9·192734
$L\tan\frac{A}{2}$	10·161683

$$\therefore\quad \frac{A}{2}=55°\,25'\,37''\cdot7$$
$$A=110°\,51'\,15''$$

Similarly *to find B.*

$$\tan\frac{B}{2}=\sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin s\sin(s-b)}}=\frac{1}{\sin(s-b)}\cdot\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}$$

$$=\frac{X}{\sin(s-b)},$$

$$\therefore\quad L\tan\frac{B}{2}=10+\log X-L\sin(s-b).$$

$$10+\log X=19\cdot354417$$
$$L\sin(s-b)=\ \ 9\cdot696370$$
$$L\tan\frac{B}{2}=\ \ 9\cdot658047$$
$$\therefore\quad \frac{B}{2}=24°\,28'\,2''$$
$$B=48°\,56'\,4''$$

also *to find C.*

$$\tan\frac{C}{2}=\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s\sin(s-c)}}=\frac{1}{\sin(s-c)}\cdot\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}$$

$$=\frac{X}{\sin(s-c)},$$

$$L\tan\frac{C}{2}=10+\log X-L\sin(s-c).$$

$$10+\log X=19\cdot354417$$
$$L\sin(s-c)=\ \ 9\cdot811977$$
$$L\tan\frac{C}{2}=\ \ 9\,542440$$
$$\frac{C}{2}=19°\,13'\,23''\cdot4$$
$$C=38°\,26'\,47''$$

Note.—After log X has been calculated the finding of the three angles is a very short process.

120. Case I. Third method—the same example.

To find A.

$$\cos\frac{A}{2}=\sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}}=\sqrt{\sin s\sin(s-a)\operatorname{cosec} b\operatorname{cosec} c},$$

$$\mathrm{L}\cos\frac{A}{2}=\tfrac{1}{2}\{\mathrm{L}\sin s+\mathrm{L}\sin(s-a)+\mathrm{L}\operatorname{cosec} b+\mathrm{L}\operatorname{cosec} c\}-10.$$

$$\begin{array}{lr}
\mathrm{L}\sin s & 9\cdot992247\\
\mathrm{L}\sin(s-a) & 9\cdot192734\\
\mathrm{L}\operatorname{cosec} b & 10\cdot119585\\
\mathrm{L}\operatorname{cosec} c & 10\cdot203295\\
\hline
 & 2)39\cdot507861\\
\mathrm{L}\cos\frac{A}{2} & 9\cdot753930\\
\hline
\frac{A}{2}= & 55^\circ\ 25'\ 37''\cdot8\\
\hline
 & =110^\circ\ 51'\ 15''
\end{array}$$

To find B.

$$\cos\frac{B}{2}=\sqrt{\frac{\sin s\sin(s-b)}{\sin a\sin c}}=\sqrt{\sin s\sin(s-b)\operatorname{cosec} a\operatorname{cosec} c},$$

$$\mathrm{L}\cos\frac{B}{2}=\tfrac{1}{2}\{\mathrm{L}\sin s+\mathrm{L}\sin(s-b)+\mathrm{L}\operatorname{cosec} a+\mathrm{L}\operatorname{cosec} c\}-10.$$

$$\begin{array}{lr}
\mathrm{L}\sin s & 9\cdot992247\\
\mathrm{L}\sin(s-b) & 9\cdot696370\\
\mathrm{L}\operatorname{cosec} a & 10\cdot026359\\
\mathrm{L}\operatorname{cosec} c & 10\cdot203295\\
\hline
 & 2)39\cdot918271\\
\mathrm{L}\cos\frac{B}{2} & 9\cdot959136\\
\hline
\frac{B}{2}= & 24^\circ\ 28'\ 2''\\
\hline
B= & 48^\circ\ 56'\ 4''
\end{array}$$

To find C.

$$\cos\frac{C}{2}=\sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}}=\sqrt{\sin s\sin(s-c)\operatorname{cosec} a\operatorname{cosec} b},$$

$$\mathrm{L}\cos\frac{C}{2}=\tfrac{1}{2}\{\mathrm{L}\sin s+\mathrm{L}\sin(s-c)+\mathrm{L}\operatorname{cosec} a+\mathrm{L}\operatorname{cosec} b\}-10.$$

$$\begin{array}{lr}
\mathrm{L}\sin s & 9\cdot992247\\
\mathrm{L}\sin(s-c) & 9\cdot811977\\
\mathrm{L}\operatorname{cosec} a & 10\cdot026359\\
\mathrm{L}\operatorname{cosec} b & 10\cdot119585\\
\hline
 & 2)39\cdot950168\\
\mathrm{L}\cos\frac{C}{2} & 9\cdot975084\\
\hline
\frac{C}{2}= & 19^\circ\ 13'\ 23''\\
\hline
C= & 38^\circ\ 26'\ 46''
\end{array}$$

Methods illustrating the use of subsidiary angles, and not requiring L haversines or tabular versed sines

121. CASE II. Given $a = 68° \ 20' \ 25''$, $b = 52° \ 18' \ 15''$, $C = 117° \ 12' \ 20''$, to find c by the aid of a subsidiary angle (see § 86).

$\overset{-}{\tan}\ \theta = \overset{+}{\tan}\ b\ \overset{-}{\cos}\ C$		$\overset{-}{\cos}\ c : \overset{+}{\cos}\ b :: \overset{+}{\cos}\ (a \sim \theta) : \overset{-}{\cos}\ \theta$	
L tan b	10·111949	L cos b	9·786375
L cos C	9·660091	L cos $(a \sim \theta)$	9·191907
		L sec θ	10·065168
L tan θ	9·772040		
		L cos c	9·043450
$180° - \theta = 30° \ 36' \ 33''$			
180			$83° \ 39' \ 16''·5$
			180
$\theta = 149° \ 23' \ 27''$			
$a = 68 \ \ 20 \ \ 25$		$c =$	$96° \ 20' \ 43''·5$
$\theta \sim a = 81° \ \ 3' \ \ 2''$		since cos c is negative.	

122. CASE III. Give the three angles to find the three sides. The supplements of the three angles will be the three corresponding sides of the polar triangle. By one of the methods of Case I. find the three angles of the polar triangle; the supplements of these three angles will be the corresponding sides of the primitive triangle.

123. CASE IV. Given two angles and the interjacent side. The supplements of these parts will be the corresponding two sides and included angle of the polar triangle. By Case II. find the third side of the polar triangle. Then with the three sides of the polar triangle find, by one of the methods of Case I., the remaining angles. The supplements of the side and two angles of the polar triangle thus found will be the required angle and two sides of the primitive triangle.

124. CASE V. Given $A = 42° \ 35' \ 30''$, $b = 61° \ 25' \ 45''$, $a = 70° 30' 15''$, to find C and c by the aid of a subsidiary angle (see § 89).

To find C.

$\tan \theta = \cot A \sec b$		$\cos (C \sim \theta) : \cos \theta :: \cot a : \cot b$	
L cot A	10·036553	L cos θ	9·604745
L sec b	10·320350	L cot a	9·549048
		L tan b	10·263956
L tan θ	10·356903		
		L cos $(C \sim \theta)$	9·417749
$\theta = 66° \ 16' \ 0''$			
		$C \sim \theta =$	$74° \ 50' \ 0''$
		$\theta =$	$66 \ \ 16 \ \ 0$
		$C =$	$141° \ 6' \ 0''$

To find c.

$\tan x = \tan b \cos A$

$L \tan b$	10·263956
$L \cos A$	9·866993
$L \tan x$	10·130949

$x = 53^\circ\ 30'\ 30''$

$\cos(c \sim x) : \cos x :: \cos a : \cos b$

$L \cos x$	9·774302
$L \cos a$	9·523406
$L \sec B$	10·320350
$L \cos(c \sim x)$	9·618058

$c \sim x =$	$65^\circ\ 28'\ 45''$
$x =$	53 30 30
$c =$	$118^\circ\ 59'\ 15''$

Note.—The values found for $C \sim \theta$ and θ, $c \sim x$ and x will only admit of one value for C and one for c.

This is what we should expect also, for since a lies between b and $180^\circ - b$ there can be only one triangle having the given parts.

125. Case V. Given $A = 137^\circ\ 24'\ 30''$, $b = 61^\circ\ 25'\ 45''$, $a = 125^\circ\ 34'\ 15''$, to find C and c by the aid of a subsidiary angle (see § 89).

To find C.

$\overset{-}{\tan\theta} = \overset{-}{\cot A}\ \overset{+}{\sec b}$

$L \cot A$	10·036553
$L \sec b$	10·320350
$L \tan\theta$	10·356903
	$66^\circ\ 16'\ 0''$
	180
$\theta =$	$113^\circ\ 44'\ 0''$

since $\tan\theta$ is negative

$\overset{+}{\cos(C \sim \theta)} : \overset{-}{\cos\theta} : : \overset{-}{\cot a} : \overset{+}{\cot b}$

$L \cos\theta$	9·604745
$L \cot a$	9·854403
$L \tan b$	10·263956
$L \cos(C \sim \theta)$	9·723104

$C \sim \theta =$	$58^\circ\ 5'\ 30''$
$\theta =$	113 44 0
$C =$	$171^\circ\ 49'\ 30''$
or $=$	55 38 30

To find c.

$\overset{-}{\tan x} = \overset{+}{\tan b}\ \overset{-}{\cos A}$

$L \tan b$	10·263956
$L \cos A$	9·866993
$L \tan x$	10·130949
	$53^\circ\ 30'\ 30''$
	180
$x =$	$126^\circ\ 29'\ 30''$

since $\tan x$ is negative

$\overset{+}{\cos(c \sim x)} : \overset{-}{\cos x} : : \overset{-}{\cos a} : \overset{+}{\cos b}$

$L \cos x$	9·774302
$L \cos a$	9·764706
$L \sec b$	10·320350
$L \cos(c \sim x)$	9·859358

$c \sim x =$	$43^\circ\ 40'\ 0''$
$x =$	126 29 30
$c =$	$170^\circ\ 9'\ 30''$
or $=$	82 49 0

Note.—The values found for $C \sim \theta$ and θ and for $c \sim x$ and x admit of two values for C and of two for c. This is what we should expect, for since a does not lie between b and $180^\circ - b$ there are two triangles which have the given parts.

126. Case VI. (1) Given $A = 42^\circ\ 35'\ 30''$, $B = 130^\circ\ 16'\ 45''$, $b = 118^\circ\ 34'\ 15''$, to find C and c by the aid of a subsidiary angle (see § 90).

To find C.

$$\overset{+}{\tan}\,\theta = -\overset{-}{\cos}\, b\, \overset{+}{\tan}\, A$$

L cos b	9·679650
L tan A	9·963447
L tan θ	9·643097

$\theta = 23^\circ\ 44'\ 0''$

$$\overset{+}{\cos}\,(C \sim \theta) : \overset{+}{\cos}\,\theta :: -\overset{-}{\cos}\, B : \overset{+}{\cos}\, A$$

L cos θ	9·961628
L cos B	9·810577
L sec A	10·133007
L cos (C $\sim \theta$)	9·905212

$C \sim \theta = 36^\circ\ 29'\ 45''$
$\theta = 23\ \ 44\ \ 0$

$C = 60^\circ\ 13'\ 45''$

To find c.

$$\overset{+}{\cot}\, x = -\overset{-}{\tan}\, b\, \overset{+}{\cos}\, A$$

L tan b	10·263956
L cos A	9·866993
L cot x	10·130949

$x = 36^\circ\ 29'\ 30''$

$$\overset{+}{\cos}\,(c \sim x) : \overset{+}{\cos}\, x :: -\overset{+}{\tan}\, A : \overset{-}{\tan}\, B$$

L cos x	9·905225
L tan A	9·963447
L cot B	9·928107
L cos ($c \sim x$)	9·796779

$c \sim x = 51^\circ\ 13'\ 15''$
$x = 36\ \ 29\ \ 30$

$c = 87^\circ\ 42'\ 45''$

Note.—The values found for $C \sim \theta$ and θ and for $c \sim x$ and x will only admit of one value for C and one for c. This we should know also from the fact that B lies between A and $180^\circ - A$, and consequently there can be only one triangle having the given parts. For three other methods of finding the angle C and the side c the student may consult §§ 111, 112, 133, where he will find the same example worked.

127. Case VI. (2) Given $A = 137^\circ\ 24'\ 30''$, $b = 118^\circ\ 34'\ 15''$, $B = 140^\circ\ 10'$, to find C and c by the aid of a subsidiary angle (see § 90).

To find C.

$$\overset{-}{\tan\theta} = -\overset{-}{\cos b}\ \overset{-}{\tan A}$$

L cos b	9·679650
L tan A	9·963447
L tan θ	9·643097
	$23^\circ\ 44'\ 0''$
	180
$\theta =$	$156^\circ\ 16'\ 0''$

since tan θ is negative

$$\overset{+}{\cos(C \sim \theta)} : \overset{-}{\cos\theta} :: -\overset{-}{\cos B} : \overset{-}{\cos A}.$$

L cos θ	9·961624
L cos B	9·885311
L sec A	10·133007
L cos $(C \sim \theta)$	9·979942
$C \sim \theta =$	$17^\circ\ 16'\ 45''$
$\theta =$	156 16 0
$C =$	$173^\circ\ 32'\ 45''$
or $=$	138 59 15

To find c.

$$\overset{-}{\cot x} = -\overset{-}{\tan b}\ \overset{-}{\cos A}$$

L tan b	10·263956
L cos A	9·866993
L cot x	10·130949
	$36^\circ\ 29'\ 30''$
	180
$x =$	$143^\circ\ 30'\ 30''$

since cot x is negative

$$\overset{+}{\cos(c \sim x)} : \overset{-}{\cos x} :: -\overset{-}{\tan A} : \overset{-}{\tan B}.$$

L cos x	9·905225
L tan A	9·963447
L cot B	10·078753
L cos $(c \sim x)$	9·947425
$c \sim x =$	$27^\circ\ 37'\ 45''$
$x =$	143 30 30
$c =$	$171^\circ\ 8'\ 15''$
or $=$	115 52 45

Note.—That there are two values for C and two for c, is clear from the values obtained for $C \sim \theta$ and θ, $c \sim x$ and x respectively, but we should also know that this would be the case since B does not lie between A and $180^\circ - A$. There are in fact two triangles having the given parts, and the two values of a are supplemental. and the two values of C and of c are as found (see also §§ 113, 114).

CHAPTER XIII

SOLUTION OF OBLIQUE-ANGLED SPHERICAL TRIANGLES

A direct method for Case III., also methods illustrative of the use of Napier's Analogies

128. CASE III. Given the three angles (A, B, C), to find the three sides (a, b, c).

Example. Given $A = 109°\ 45'\ 40''$, $B = 130°\ 35'\ 50''$, $C = 141°\ 13'\ 50''$, find (a, b, c) (see § 91).

$A = 109°\ 45'\ 40''$	$A = 109°\ 45'\ 40''$	$B = 130°\ 35'\ 50''$	$C = 141°\ 13'\ 50''$
$B = 130\ \ 35\ \ 50$	$\frac{E}{2} = 100\ \ 47\ \ 40$	$\frac{E}{2} = 100\ \ 47\ \ 40$	$\frac{E}{2} = 100\ \ 47\ \ 40$
$C = 141\ \ 13\ \ 50$			
$381\ \ 35\ \ 20$	$A - \frac{E}{2} = 8°\ 58'\ 0''$	$B - \frac{E}{2} = 29°\ 48'\ 10''$	$C - \frac{E}{2} = 40°\ 26'\ 10''$
180			
$E = 201\ \ 35\ \ 20$			
$2A = 219\ \ 31\ \ 20$			
$2A - E = 17°\ 56'\ 0''$			

To find a.

(1)
$$\text{hav } a = \text{cosec B cosec C} \sqrt{\text{hav E hav (2A} - \text{E)}},$$
$$\text{L hav } a = \text{L cosec B} + \text{L cosec C} + \tfrac{1}{2}\text{L hav E} + \tfrac{1}{2}\text{L hav (2A} - \text{E)} - 20.$$

$\frac{1}{2}$L hav E	4·992246
$\frac{1}{2}$L hav (2A − E)	4·192734
L cosec B	10·119585
L cosec C	10·203295
L hav a	9·507860
$a =$	$69°\ 8'\ 45''$

(2)
$$\sin\frac{a}{2} = \sqrt{\frac{\sin\frac{E}{2}\sin\left(A - \frac{E}{2}\right)}{\sin B \sin C}}$$

$$\text{L} \sin\frac{a}{2} = \tfrac{1}{2}\left\{\text{L} \sin\frac{E}{2} + \text{L} \sin\left(A - \frac{E}{2}\right) + \text{L cosec B} + \text{L cosec C}\right\} - 10.$$

$L \sin \frac{E}{2}$	9·992249
$L \sin \left(A-\frac{E}{2}\right)$	9·192734
L cosec B	10·119585
L cosec C	10·203295
	2)39·507863
$L \sin \frac{a}{2}$	9·753931
$\frac{a}{2}=$	34° 34′ 22″·5
$a=$	69° 8′ 45″

To find b.

$$\cos \frac{b}{2}=\sqrt{\frac{\sin \left(A-\frac{E}{2}\right) \sin \left(C-\frac{E}{2}\right)}{\sin A \sin C}}$$

$$L \cos \frac{b}{2}=\tfrac{1}{2}\left\{L \sin \left(A-\frac{E}{2}\right)+L \sin \left(C-\frac{E}{2}\right)+L \operatorname{cosec} A+L \operatorname{cosec} C\right\}-10.$$

$L \sin \left(A-\frac{E}{2}\right)$	9·192734
$L \sin \left(C-\frac{E}{2}\right)$	9·811977
L cosec A	10·026359
L cosec C	10·203295
	2)39·234365
$L \cos \frac{b}{2}$	9·617183
$\frac{b}{2}=$	65° 31′ 58″
$b=$	131° 3′ 56″

To find c.

$$\tan \frac{c}{2}=\sqrt{\frac{\sin \frac{E}{2} \sin \left(C-\frac{E}{2}\right)}{\sin \left(A-\frac{E}{2}\right) \sin \left(B-\frac{E}{2}\right)}}$$

$$L \tan \frac{c}{2}=\tfrac{1}{2}\left\{L \sin \frac{E}{2}+L \sin \left(C-\frac{E}{2}\right)-L \sin \left(A-\frac{E}{2}\right)-L \sin \left(B-\frac{E}{2}\right)\right\}+10.$$

$L \sin \frac{E}{2}$	9·992249	$L \sin \left(A-\frac{E}{2}\right)$	9·192734
$L \sin \left(C-\frac{E}{2}\right)$	9·811977	$L \sin \left(B-\frac{E}{2}\right)$	9·696371
	19·804226		18·889105
	18·889105		
	2)0·915121		
$L \tan \frac{c}{2}$	10·457560		
$\frac{c}{2}=$	70° 46′ 36″·5		
$c=$	141° 33′ 13″		

These examples illustrate how any *one* side may be found when the three angles are given.

129. If all *three* sides are to be found the following method is shorter.

To find a.

$$\tan\frac{a}{2}=\sin\left(A-\frac{E}{2}\right)\sqrt{\frac{\sin\frac{E}{2}}{\sin\left(A-\frac{E}{2}\right)\sin\left(B-\frac{E}{2}\right)\sin\left(C-\frac{E}{2}\right)}}$$

$$=\sin\left(A-\frac{E}{2}\right)\times X \text{ (see § 91)},$$

$$L\tan\frac{a}{2}=\tfrac{1}{2}\left\{L\sin\frac{E}{2}-L\sin\left(A-\frac{E}{2}\right)-L\sin\left(B-\frac{E}{2}\right)-L\sin\left(C-\frac{E}{2}\right)\right\}$$
$$+L\sin\left(A-\frac{E}{2}\right)+10$$

$$=10+\log X+L\sin\left(A-\frac{E}{2}\right).$$

Similarly

$$L\tan\frac{b}{2}=10+\log X+L\sin\left(B-\frac{E}{2}\right)$$

and

$$L\tan\frac{c}{2}=10+\log X+L\sin\left(C-\frac{E}{2}\right).$$

Using the same example as in § 128

$L\sin\left(A-\frac{E}{2}\right)$	9·192734
$L\sin\left(B-\frac{E}{2}\right)$	9·696371
$L\sin\left(C-\frac{E}{2}\right)$	9·811977
	28·701082
$L\sin\frac{E}{2}$	9·992249
	2)19·291167
$10+\log X$	0·645583
$L\sin\left(A-\frac{E}{2}\right)$	9·192734
$L\tan\frac{a}{2}$	9·838317
$\frac{a}{2}=$	34° 34′ 22″·3
$a=$	69° 8′ 45″

$10+\log X$	0·645583
$L\sin\left(B-\frac{E}{2}\right)$	9·696371
$L\tan\frac{b}{2}$	10·341954
$\frac{b}{2}=$	65° 31′ 58″
$b=$	131° 3′ 56″

$10+\log X$	0·645583
$L\sin\left(C-\frac{E}{2}\right)$	9·811977
$L\tan\frac{c}{2}$	10·457560
$\frac{c}{2}=$	70° 46′ 36″·5
$c=$	141° 33′ 13″

Methods illustrative of the use of Napier's Analogies (see § 94).

130. Case II. Given two sides and the included angle (a, b, C), to find the other two angles and the third side (A, B, c) (see § 95).

Example. Given $a = 68^\circ\ 20'\ 25''$, $b = 52^\circ\ 18'\ 15''$, $C = 117^\circ\ 12'\ 20''$.

To find A, B.

By Napier's Analogies,

$$\tan \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \frac{C}{2} \qquad \tan \tfrac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cot \frac{C}{2}.$$

$a = 68^\circ\ 20'\ 25''$
$b = 52\ \ 18\ \ 15$

$a - b = 16\ \ 2\ \ 10$
$a + b = 120\ \ 38\ \ 40$

$\frac{1}{2}(a-b) = 8\ \ 1\ \ 5$
$\frac{1}{2}(a+b) = 60\ \ 19\ \ 20$

$\frac{C}{2} = 58^\circ\ 36'\ 10''$

$L \cos \frac{1}{2}(a-b)$	9·995734	$L \sin \frac{1}{2}(a-b)$	9·144528
$L \sec \frac{1}{2}(a+b)$	10·305288	$L \operatorname{cosec} \frac{1}{2}(a+b)$	10·061068
$L \cot \frac{C}{2}$	9·785569	$L \cot \frac{C}{2}$	9·785569
$L \tan \frac{1}{2}(A+B)$	10·086591	$L \tan \frac{1}{2}(A-B)$	8·991165
		$\frac{1}{2}(A-B)$	$5^\circ\ 35'\ 47''$

$\frac{1}{2}(A+B) = 50^\circ\ 40'\ 28''$
$\frac{1}{2}(A-B) = 5\ \ 35\ \ 47$

$A = 56^\circ\ 16'\ 15''$
$B = 45\ \ 4\ \ 41$

To find c.

By Napier's Analogy,

$$\tan \frac{c}{2} = \frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)} \tan \tfrac{1}{2}(a+b).$$

$L \cos \frac{1}{2}(A+B)$	9·801901
$L \sec \frac{1}{2}(A-B)$	10·002075
$L \tan \frac{1}{2}(a+b)$	10·244220
$L \tan \frac{c}{2}$	10·048196
$\frac{c}{2}$	$48^\circ\ 10'\ 22''$
c	$96^\circ\ 20'\ 44''$

131. Case IV. Given two angles and the interjacent side (A, B, c), to find the remaining two sides and the remaining angle (a, b, C) (see § 96).

Example. Given A = 111° 39′ 35″, B = 127° 41′ 45″,
c = 62° 47′ 40″.

To find a, b.

By Napier's Analogies,

$$\tan\tfrac{1}{2}(b+a)=\frac{\cos\frac{1}{2}(B-A)}{\cos\frac{1}{2}(B+A)}\tan\frac{c}{2}\qquad \tan\tfrac{1}{2}(b-a)=\frac{\sin\frac{1}{2}(B-A)}{\sin\frac{1}{2}(B+A)}\tan\frac{c}{2}.$$

A = 111° 39′ 35″	$L\cos\frac{1}{2}(B-A)$	9·995734	$L\sin\frac{1}{2}(B-A)$	9·144528
B = 127 41 45	$L\sec\frac{1}{2}(B+A)$	10·305288	$L\operatorname{cosec}\frac{1}{2}(B+A)$	10·061068
B − A = 16 2 10	$L\tan\frac{c}{2}$	9·785569	$L\tan\frac{c}{2}$	9·785569
B + A = 239 21 20				
$\frac{B-A}{2}$ = 8 1 5	$L\tan\frac{1}{2}(b+a)$	10·086591	$L\tan\frac{1}{2}(b-a)$	8·991165
$\frac{B+A}{2}$ = 119 40 40	$180^\circ-\frac{1}{2}(b+a)=$	50° 40′ 28″	$\frac{1}{2}(b-a)=$	5° 35′ 47″
	$\therefore\ \frac{1}{2}(b+a)=$	129 19 32		
	$\frac{1}{2}(b-a)=$	5 35 47		
$\frac{c}{2}$ = 31° 23′ 50″	$a=$	123° 43′ 45″		
	$b=$	134 55 19		

To find C.

By Napier's Analogy,

$$\cot\frac{C}{2}=\frac{\cos\frac{1}{2}(b+a)}{\cos\frac{1}{2}(b-a)}\tan\tfrac{1}{2}(B+A).$$

$L\cos\frac{1}{2}(b+a)$	9·801896
$L\sec\frac{1}{2}(b-a)$	10·002075
$L\tan\frac{1}{2}(B+A)$	10·244220
$L\cot\frac{C}{2}$	10·048191
$\frac{C}{2}=$	41° 49′ 39″·4
$C=$	83° 39′ 19″

132. Case V. Given two sides and an angle opposite one of them (a, b, A), to find (B, C, c) (see § 97).

Example. Given A = 42° 35′ 30″, a = 70° 30′ 15″, b = 61° 25′ 45″.

Here a lies between b and $180^\circ - b$, hence there is only one triangle having the given parts and B will be of like affection with b.

To find B.

By Rule of Sines,

$$\sin B = \frac{\sin b}{\sin a}\sin A.$$

L sin b	9·943607
L cosec a	10·025642
L sin A	9·830440
L sin B	9·799689

$$B = 39^\circ\ 5'\ 15''$$

To find C and c.

By Napier's Analogies,

$$\tan\frac{C}{2} = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\tfrac{1}{2}(A+B) \qquad \tan\frac{c}{2} = \frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)}\tan\tfrac{1}{2}(a+b).$$

$a=$	70° 30′ 15″
$b=$	61 25 45
$a-b=$	9 4 30
$a+b=$	131 56 0
$\frac{a-b}{2}=$	4 32 15
$\frac{a+b}{2}=$	65 58 0
A $=$	42 35 30
B $=$	39 5 15
A $-$ B $=$	3 30 15
A $+$ B $=$	81 40 45
$\frac{A-B}{2}=$	1° 45′ 7″·5
$\frac{A+B}{2}=$	40 50 22·5

L cos $\frac{1}{2}(a-b)$	9·998637
L sec $\frac{1}{2}(a+b)$	10·390120
L cot $\frac{1}{2}(A+B)$	10·063294
L tan $\frac{C}{2}$	10·452051
$\frac{C}{2}=$	70° 32′ 48″
C $=$	141° 5′ 36″

L cos $\frac{1}{2}(A+B)$	9·878833
L sec $\frac{1}{2}(A-B)$	10·000203
L tan $\frac{1}{2}(a+b)$	10·350737
L tan $\frac{c}{2}$	10·229773
$\frac{c}{2}=$	59° 29′ 44″
$c=$	118° 59′ 28″

133. Case VI. Given two angles and a side opposite one of these angles (A, B, b), to find the other two sides and the remaining angle (see § 98).

Example. Given $A = 42^\circ\ 35'\ 30''$, $B = 130^\circ\ 16'\ 45''$,
$b = 118^\circ\ 34'\ 15''$.

Here B lies between A and $180^\circ - A$, hence there is only one triangle having the given parts, and a will be of like affection with A.

To find a.

By Rule of Sines,

$$\sin a = \frac{\sin A}{\sin B}\sin b.$$

$$\begin{aligned} L \sin A &= 9{\cdot}830440 \\ L \operatorname{cosec} B &= 10{\cdot}117530 \\ L \sin b &= 9{\cdot}943607 \\ \hline L \sin a &= 9{\cdot}891577 \\ \hline a &= 51^\circ\ 10'\ 32'' \end{aligned}$$

To find C and c.

By Napier's Analogies,

$$\tan\frac{C}{2} = \frac{\cos\frac{1}{2}(b-a)}{\cos\frac{1}{2}(b+a)}\cot\tfrac{1}{2}(B+A) \qquad \tan\frac{c}{2} = \frac{\cos\frac{1}{2}(B+A)}{\cos\frac{1}{2}(B-A)}\tan\tfrac{1}{2}(b+a).$$

$$\begin{aligned} b &= 118^\circ\ 34'\ 15'' \\ a &= 51\ \ 10\ \ 32 \\ \hline b-a &= 67\ \ 23\ \ 43 \\ b+a &= 169\ \ 44\ \ 47 \\ \hline \tfrac{1}{2}(b-a) &= 33\ \ 41\ \ 51{\cdot}5 \\ \tfrac{1}{2}(b+a) &= 84\ \ 52\ \ 23{\cdot}5 \\ \hline B &= 130\ \ 16\ \ 45 \\ A &= 42\ \ 35\ \ 30 \\ \hline B-A &= 87\ \ 41\ \ 15 \\ B+A &= 172\ \ 52\ \ 15 \\ \hline \tfrac{1}{2}(B-A) &= 43^\circ\ 50'\ 37''{\cdot}5 \\ \tfrac{1}{2}(B+A) &= 86\ \ 26\ \ \ 7{\cdot}5 \end{aligned}$$

$$\begin{aligned} L \cos\tfrac{1}{2}(b-a) &\quad 9{\cdot}920111 \\ L \sec\tfrac{1}{2}(b+a) &\quad 11{\cdot}048855 \\ L \cot\tfrac{1}{2}(B+A) &\quad 8{\cdot}794447 \\ \hline L \tan\frac{C}{2} &\quad 9{\cdot}763413 \\ \hline \frac{C}{2} &= 30^\circ\ 6'\ 46''{\cdot}5 \\ \hline C &= 60^\circ\ 13'\ 33'' \end{aligned}$$

$$\begin{aligned} L \cos\tfrac{1}{2}(B+A) &\quad 8{\cdot}793606 \\ L \sec\tfrac{1}{2}(B-A) &\quad 10{\cdot}141926 \\ L \tan\tfrac{1}{2}(b+a) &\quad 11{\cdot}047115 \\ \hline L \tan\frac{c}{2} &\quad 9{\cdot}982647 \\ \hline \frac{c}{2} &= 43^\circ\ 51'\ 20'' \\ \hline c &= 87^\circ\ 42'\ 40'' \end{aligned}$$

EXAMPLES

I

1. $a=96^\circ\ 24'\ 30''$, $b=68^\circ\ 27'\ 26''$, $c=87^\circ\ 31'\ 37''$.
2. $a=102^\circ\ 21'\ 42''$, $b=78^\circ\ 17'\ 2''$, $c=126^\circ\ 46'\ 0''$.
3. $a=148^\circ\ 25'\ 34''$, $b=149^\circ\ 31'\ 48''$, $c=49^\circ\ 56'\ 10''$.
4. $a=83^\circ\ 13'\ 30''$, $b=95^\circ\ 29'\ 40''$, $c=54^\circ\ 31'\ 47''$.
5. $a=b=119^\circ\ 6'\ 58''$, $c=113^\circ\ 10'\ 43''$.
6. $a=85^\circ\ 16'$, $b=63^\circ\ 24'$, $c=75^\circ\ 33'\ 8''$.
7. $a=85^\circ\ 56'\ 10''$, $b=105^\circ\ 36'\ 15''$, $c=116^\circ\ 26'\ 16''$.
8. $a=100^\circ\ 29'\ 15''$, $b=133^\circ\ 44'\ 45''$, $c=109^\circ\ 4'\ 25''$.
9. $a=133^\circ\ 29'\ 20''$, $b=143^\circ\ 39'\ 40''$, $c=69^\circ\ 56'\ 45''$.
10. $a=95^\circ\ 29'\ 40''$, $b=103^\circ\ 39'\ 20''$, $c=49^\circ\ 8'\ 26''$.

II

1. $a=82^\circ\ 7'\ 0''$, $b=112^\circ\ 0'\ 21''$, $C=92^\circ\ 28'\ 23''$.
2. $a=133^\circ\ 46'\ 30''$, $b=113^\circ\ 9'\ 40''$, $C=98^\circ\ 48'\ 0''$.
3. $a=111^\circ\ 20'\ 0''$, $b=123^\circ\ 40'\ 0''$, $C=95^\circ\ 30'\ 0''$.
4. $a=158^\circ\ 58'\ 58''$, $b=171^\circ\ 21'\ 14''$, $C=103^\circ\ 38'\ 0''$.
5. $a=93^\circ\ 29'\ 20''$, $b=125^\circ\ 18'\ 46''$, $C=126^\circ\ 34'\ 0''$.
6. $b=105^\circ\ 40'\ 0''$, $c=96^\circ\ 50'\ 0''$, $A=114^\circ\ 38'\ 37''$.
7. $a=56^\circ\ 6'\ 13''$, $c=133^\circ\ 8'\ 9''$, $B=104^\circ\ 31'\ 0''$.
8. $a=148^\circ\ 25'\ 34''$, $b=149^\circ\ 31'\ 48''$, $C=109^\circ\ 57'\ 57''$.
9. $c=63^\circ\ 29'\ 32''$, $b=128^\circ\ 13'\ 48''$, $A=103^\circ\ 35'\ 30''$.
10. $a=114^\circ\ 46'\ 56''$, $c=51^\circ\ 44'\ 0''$, $B=123^\circ\ 12'\ 0''$.

III

1. $A=95^\circ\ 45'\ 31''$, $B=135^\circ\ 46'\ 15''$, $C=128^\circ\ 53'\ 48''$.
2. $A=93^\circ\ 41'\ 20''$, $B=134^\circ\ 23'\ 40''$, $C=71^\circ\ 20'\ 48''$.
3. $A=136^\circ\ 42'\ 0''$, $B=160^\circ\ 36'\ 0''$, $C=138^\circ\ 24'\ 11''$.
4. $A=133^\circ\ 36'\ 0''$, $B=112^\circ\ 46'\ 0''$, $C=98^\circ\ 48'\ 0''$.
5. $A=133^\circ\ 14'\ 30''$, $B=138^\circ\ 7'\ 26''$, $C=138^\circ\ 24'\ 56''$.
6. $A=145^\circ\ 30'\ 0''$, $B=164^\circ\ 29'\ 30''$, $C=140^\circ\ 26'\ 8''$.
7. $A=120^\circ\ 31'\ 33''$, $B=131^\circ\ 20'\ 44''$, $C=109^\circ\ 36'\ 18''$.
8. $A=111^\circ\ 15'\ 0''$, $B=126^\circ\ 45'\ 0''$, $C=133^\circ\ 30'\ 0''$.
9. $A=83^\circ\ 35'\ 30''$, $B=111^\circ\ 32'\ 34''$, $C=92^\circ\ 28'\ 23''$.
10. $A=110^\circ\ 48'\ 12''$, $B=123^\circ\ 21'\ 30''$, $C=95^\circ\ 30'\ 0''$.

IV

1. $A=121° 26' 22''$, $C=104° 6' 0''$, $b=138° 32'$.
2. $B=116° 36' 0''$, $C=104° 26' 52''$, $a=88° 4' 3''$.
3. $A=98° 13' 53''$, $B=122° 16' 16''$, $c=97° 3' 56''$.
4. $A=103° 24' 24''$, $C=139° 59' 50''$, $b=137° 44' 47''$.
5. $A=44° 10' 40''$, $B=33° 22' 45''$, $c=74° 51' 50''$.
6. $A=121° 36' 20''$, $C=34° 15' 3''$, $b=50° 10' 30''$.
7. $A=109° 45' 40''$, $B=130° 35' 50''$, $c=141° 33' 12''$.
8. $B=140° 32' 0''$, $C=119° 39' 49''$, $a=97° 35' 27''$.
9. $A=95° 30' 0''$, $B=123° 14' 0''$, $c=106° 8' 0''$.
10. $A=122° 52' 53''$, $B=110° 37' 40''$, $c=120° 14' 0''$.

V

1. $A=126° 41' 40''$, $a=100° 29' 15''$, $b=133° 44' 45''$.
2. $A=133° 14' 30''$, $a=106° 48' 42''$, $b=118° 41' 48''$.
3. $A=95° 56' 18''$, $a=84° 30' 0''$, $c=75° 56' 27''$.
4. $A=58° 16' 0''$, $a=58° 33' 37''$, $b=41° 47' 40''$.
5. $C=82° 56' 4''$, $c=79° 40' 0''$, $b=57° 43' 44''$.
6. $B=36° 17' 21''$, $b=36° 22' 0''$, $c=56° 21' 30''$.
7. $B=74° 23' 45''$, $b=74° 35' 0''$, $c=63° 40' 0''$.
8. $C=137° 44' 40''$, $c=133° 29' 20''$, $b=143° 39' 40''$.
9. $A=137° 44' 40''$, $a=143° 39' 40''$, $b=133° 29' 20''$.
10. $C=120° 31' 33''$, $c=127° 9' 40''$, $b=113° 52' 40''$.
11. $A=70° 22' 16''$, $a=56° 48' 0''$, $b=42° 26' 0''$.
12. $A=75° 18' 0''$, $a=60° 20' 10''$, $b=39° 28' 0''$.

VI

1. $B=59° 51' 11''$, $C=79° 41' 48''$, $c=86° 40' 0''$.
2. $A=63° 48' 35''$, $C=51° 46' 12''$, $a=76° 24' 40''$.
3. $C=50° 50' 52''$, $B=58° 56' 10''$, $b=53° 15' 0''$.
4. $A=64° 48' 54''$, $C=120° 46' 30''$, $a=86° 18' 40''$.
5. $C=130° 5' 22''$, $A=36° 45' 26''$, $c=84° 14' 29''$.
6. $B=108° 30' 0''$, $A=96° 45' 0''$, $a=88° 27' 49''$.
7. $A=137° 15' 0''$, $B=132° 45' 0''$, $a=123° 30' 0''$.
8. $A=117° 44' 36''$, $B=76° 41' 13''$, $a=126° 17' 22''$.
9. $A=127° 14' 40''$, $B=108° 47' 20''$, $a=133° 37' 50''$.
10. $A=B=38° 23' 57''$, $a=60° 53' 2''$.
11. $A=131° 14' 20''$, $B=112° 47' 40''$, $a=137° 39' 30''$.
12. $C=133° 39' 15''$, $B=114° 41' 45''$, $c=139° 49' 30''$.

VII

1. $C=90°$, $b=34° 36' 11''$, $a=38° 50'$.
2. $C=90°$, $A=41° 56'$, $B=70° 19'$.
3. $C=90°$, $a=48° 30'$, $b=59° 28' 0''$.
4. $C=90°$, $a=45° 30'$, $c=66° 30' 31''$.

5. $A=90°$, $C=51° \, 5' \, 47''$, $b=64° \, 30' \, 0''$.
6. $A=90°$, $c=29° \, 26' \, 30''$, $b=84° \, 37' \, 15''$.
7. $C=90°$, $a=141° \, 10' \, 0''$, $c=129° \, 52' \, 47''$.
8. $B=90°$, $c=137° \, 3' \, 48''$, $A=147° \, 2' \, 54''$.
9. $A=90°$, $b=50° \, 30' \, 29''$, $c=40° \, 31' \, 20''$.
10. $C=90°$, $A=127° \, 17' \, 51''$, $c=109° \, 40' \, 20''$.
11. $C=90°$, $B=113° \, 49' \, 31''$, $c=70° \, 19' \, 40''$.
12. $A=90°$, $C=51° \, 50'$, $c=40° \, 45'$.
13. $A=90°$, $C=66° \, 7' \, 20''$, $c=59° \, 28' \, 27''$.
14. $B=90°$, $A=62° \, 12'$, $a=51° \, 20' \, 0''$.

VIII

1. $A=141° \, 10'$, $b=132° \, 16'$, $c=90°$.
2. $A=134° \, 30'$, $b=116° \, 15'$, $c=90°$.
3. $C=115° \, 30'$, $B=131° \, 48'$, $a=90°$.
4. $a=150° \, 27'$, $c=92° \, 39'$, $b=90°$.
5. $C=139° \, 15'$, $c=128° \, 10'$, $a=90°$.
6. $a=124° \, 27' \, 15''$, $C=81° \, 45' \, 36''$, $c=90°$.
7. $C=118° \, 55' \, 4''$, $A=139° \, 28' \, 40''$, $c=90°$.
8. $C=128° \, 40'$, $c=117° \, 48'$, $a=90°$.
9. $A=132° \, 15'$, $B=110° \, 30'$, $b=90°$.
10. $B=131° \, 15'$, $a=158° \, 33'$, $b=90°$.
11. $a=133° \, 15'$, $C=104° \, 20'$, $c=90°$.
12. $b=132° \, 15'$, $a=116° \, 30'$, $c=90°$.
13. $A=93° \, 15'$, $B=71° \, 40'$, $c=90°$.
14. $C=137° \, 41' \, 15''$, $c=133° \, 44' \, 35''$, $a=90°$.

IX

1. $A=109° \, 45' \, 40''$, $b=131° \, 3' \, 56''$, $c=141° \, 33' \, 12''$.
2. $a=123° \, 43' \, 45''$, $b=134° \, 55' \, 19''$, $c=62° \, 47' \, 40''$.
3. $A=129° \, 14' \, 40''$, $B=110° \, 47' \, 20''$, $a=135° \, 37' \, 50''$.
4. $A=142° \, 11' \, 48''$, $b=109° \, 40' \, 45''$, $c=90°$.
5. $A=108° \, 24' \, 44''$, $B=139° \, 26' \, 48''$, $C=95° \, 14' \, 0''$.
6. $a=123° \, 2' \, 0''$, $b=140° \, 34' \, 0''$, $C=111° \, 12' \, 0''$.
7. $A=41° \, 55' \, 45''$, $B=70° \, 19' \, 15''$, $C=90°$.
8. $a=86° \, 45' \, 0''$, $B=108° \, 18' \, 21''$, $C=90°$.
9. $a=47° \, 45' \, 0''$, $B=65° \, 41' \, 37''$, $C=90°$.
10. $A=91° \, 24' \, 19''$, $B=67° \, 39' \, 46''$, $C=54° \, 59' \, 22''$.
11. $A=123° \, 12' \, 0''$, $B=137° \, 34' \, 0''$, $c=96° \, 48' \, 0''$.
12. $b=104° \, 40' \, 0''$, $c=117° \, 16' \, 0''$, $A=143° \, 50' \, 47''$.
13. $a=110° \, 4' \, 0''$, $b=133° \, 24' \, 0''$, $c=85° \, 38' \, 20''$.
14. $A=92° \, 33' \, 0''$, $B=116° \, 41' \, 0''$, $a=85° \, 3' \, 23''$.
15. $A=104° \, 48' \, 0''$, $B=116° \, 33' \, 0''$, $C=127° \, 37' \, 0''$.
16. $A=95° \, 30' \, 0''$, $b=133° \, 52' \, 44''$, $c=104° \, 14' \, 0''$.
17. $A=107° \, 16' \, 0''$, $B=143° \, 38' \, 0''$, $c=123° \, 48' \, 0''$.
18. $A=47° \, 45' \, 0''$, $a=41° \, 17' \, 48''$, $C=90°$.
19. $A=21° \, 27' \, 0''$, $a=15° \, 57' \, 30''$, $C=90°$.
20. $A=135° \, 6' \, 57''$, $a=133° \, 15' \, 0''$, $c=90°$.

X

1. The diameter of a sphere is 74 inches and the distance of the centre of a small circle from the centre of the sphere is 12 inches, find the radius of the small circle.

2. The radius of a small circle of a sphere is 4 inches and the radius of the sphere is 5 inches, find the distance between the centre of the sphere and the centre of the small circle.

3. The radius of a sphere is 17 inches and the radius of a small circle of the sphere is 8 inches, find the distance of the pole of the small circle from any point in its circumference.

4. Find the length of an arc of 30° of a small circle on a sphere whose radius is 13 inches ; the distance between the centre of the sphere and the centre of the small circle being 5 inches. $\pi = \frac{22}{7}$.

5. The distance between the centre of a sphere and the centre of a small circle is 9 inches, the radius of the sphere being 15 inches. Compare the lengths of an arc of 50° on each circle.

6. The arc of a great circle joining A, B upon a certain sphere is $7\frac{1}{3}$ inches in length and contains 60°, find the length of the diameter of the sphere and of the chord joining A and B. $\pi = \frac{22}{7}$.

7. The distance of the centre of a small circle from the centre of the sphere is 30 inches. The radius of the sphere being 34 inches, find the distance of the pole of the small circle from any point in its circumference.

8. On a sphere of radius 29 inches two secondaries intercept arcs of a small circle and of a great circle. The distance between the centre of the small circle and the centre of the sphere is 21 inches, find the number of degrees in the arc of a secondary between the circles.

9. If the earth were a sphere 7980 miles in diameter how many miles would be passed over in travelling between two places which are in the same latitude, 60°, and whose longitudes differ by 12° ?

10. On a sphere whose radius is 15 inches two secondaries intercept two arcs of small circles and an arc of a great circle. The centre of one of the small circles is 9 inches from the centre of the sphere and the arc of a secondary between the pole and the other small circle is 30°. Compare the intercepted arcs of the small and great circles.

11. A ship sails along the parallel of 45° N. a distance of 400 nautical miles, find the difference of longitude she has made.

12. Two places in latitude 45° N. are 150 statute miles apart, the radius of the earth being 3990 statute miles ; find their difference of longitude.

13. Compare the lengths of the parallels of 60° N., 45° N., 30° N. with the length of the equator.

14. A ship sailing on a great circle crosses the equator in 20° W. longitude, her course is then N. 40° E. By the aid of spherical geometry find the latitude and longitude of the vertex of the great circle.

15. Prove that two circles of the sphere cannot bisect one another unless they are great circles.

16. If the diameters of two small circles subtend at the centre of the sphere angles of 60° and 90°, compare those arcs of the small circles which subtend equal angles at their respective centres.

XI

1. A solid angle is contained by three plane angles 60°, 80°, 40°, find the angle between the planes of the angles 60° and 80°.

2. Two of the three angles which contain a solid angle are 30° and 60° and their planes are inclined at an angle of 45°; find the angle of the third plane face and the angles at which this third plane is inclined to the other two planes.

3. The two planes ABC, ABD intersect in AB. The lines BC and BD are at right angles to one another, and the angles ABC, ABD are 75° and 65° respectively; find the inclination of the planes.

4. The three legs of a tripod make angles at the socket 70°, 65°, 90°. A cord connecting the third leg with the two at right angles is then observed to be fully stretched and to make right angles with the direction of the third leg; find the angle contained by the cord at the third leg.

5. A pyramid has each of its slant sides and base an equilateral triangle; find the angle between any two faces.

6. A pyramid each of whose slant faces is an equilateral triangle has a square base; find the angle between any two slant faces, also the angle between any slant face and the base.

7. An equilateral triangle is inscribed in a small circle of a sphere. The radius of the small circle is 4 inches and of the sphere 8 inches. The centre of the sphere is joined to the angular points of the triangle thus forming a pyramid.

Find (*a*) The angle of a plane face of the solid angle at the centre of the sphere,
(*b*) the angle of a plane face of the solid angle at one of the points of the base,
(*c*) the angle at which any two slant faces are inclined,
(*d*) the angle at which any slant face is inclined to the base.

8. The square ABCD is inscribed in a sphere whose centre is O. If each side of the square be a and the diameter of the sphere be d, prove that the cosine of the acute angle between the planes OAB, OBC is $\frac{a^2}{d^2-a^2}$.

9. From A one corner of a cube, AG, AH, each 1 inch, are measured along two edges. From H, G to a point F on the other edge the length is 2 inches. A plane section is then made through the three points G, F, H. Show that the cosine of the angle between the planes AFH, GFH is $\sqrt{\frac{3}{7}}$.

10. A pyramid stands on an irregular polygonal base. If one face be an equilateral triangle inclined at an angle of 45° to this base, what must be the inclination of the next face, which is a right-angled isosceles triangle? and what will be the angle between the two faces?

11. If in a spherical triangle $b+c=90°$, prove that $\cos a=\sin 2c \cos^2 \frac{A}{2}$.

12. In a spherical triangle $A=1° 7'$, $b=37° 48'$, $c=38° 15'$. Find a. Also find the length of the arc a if the radius be 4000 miles.

13. If A be one of the base angles of an isosceles spherical triangle whose vertical angle is 90° and a the opposite side, prove that $\cos a=\cot A$; and determine the limits between which it is necessary that A must lie.

14. Of a spherical triangle two sides are given, prove that the angle opposite the smaller of them will be the greatest when that opposite the larger is a right angle.

XII

1. In a spherical triangle prove any two sides are greater than the third. Hence, by the aid of the polar triangle, show that the three angles of a right-angled spherical triangle must be less than 360°.

2. If a be the side of an equilateral triangle and a' that of its polar triangle, prove $\cos a \,.\, \cos a' = \frac{1}{2}$.

3. If the three angular points of a spherical triangle are on a small circle, and two of them on a diameter of it, two of its angles are together equal to the third.

4. Prove $\cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}$

What does this formula become when
   (1) $A = 90°$,
   (2) a = quadrant,
   (3) when applied to the polar triangle?

5. ABC is a spherical triangle of which each side is a quadrant and P is a point within it; prove that

$$\cos^2 PB + \cos^2 PC = \sin^2 PA.$$

Hence if $BP = 60°$ and $AP = 40°$, find APB.

6. The three angles of a spherical triangle are known. Can we from these data determine the angular measure of the arcs which form the sides of the triangle? Are they sufficient to enable us to determine the actual lengths of these arcs?

7. The side of an equilateral triangle is 60°. Completely solve the triangle, using only one logarithm in doing so.

8. Show how to divide the portion of a sphere contained between two great semicircles into two oblique isosceles triangles. Under what circumstances is the problem impossible?

9. Two sides of a spherical triangle are quadrants and the radius of the small circle inscribed in the triangle is 30°; find the cosine of the angle included by the quadrants.

10. In a spherical triangle ABC, having given $\cos a = \frac{7}{9}$, $\cos b = \frac{7}{10}$ $\cos c = \frac{7}{11}$, find cos B.

11. In an equilateral spherical triangle, prove

$$2 \sin \frac{A}{2} \cos \frac{a}{2} = 1.$$

Hence show that such a triangle can never have its angles less than 60°, nor its side greater than 120°.

12. In an equilateral spherical triangle $2 \cos A = 1 - \tan^2 \dfrac{a}{2}$.

13. In a right-angled spherical triangle prove

$$\cos c = \cos a \cos b.$$

If $\tan b = 1$, $\tan a = 2$, show that $\cos c = \dfrac{1}{\sqrt{10}}$.

14. Two angles of a spherical triangle are 130° and 50°. Show that the third angle must be less than 100°.

15. If a, b, c be the sides of a spherical triangle, and if the arc x be drawn from the angle A to bisect the opposite side then

$$\cos\frac{a}{2}\cos x=\cos\frac{b+c}{2}\cos\frac{b-c}{2}.$$

16. ABC is an isosceles triangle in which each of the equal sides is double of the third side a. Show that

$$\operatorname{cosec}\frac{A}{2}=4\cos a\cos\frac{a}{2}.$$

17. In a right-angled spherical triangle prove that $\cos A=\cot c\tan b$; hence show that

$$\tan^2\frac{A}{2}=\frac{\sin(c-b)}{\sin(c+b)}.$$

18. Show that any side of a spherical triangle is greater than the difference of the other two, and hence, by the aid of the polar triangle, show that any angle of a spherical triangle is less than the supplement of the difference of the other two. Two angles of a triangle are 80° and 70°, show that the third angle is less than 170°.

19. What will be the first course and what distance will be passed over in sailing on a great circle between two places, in latitude 45°, whose longitudes differ by 90°?

20. A spherical triangle is right-angled at C. If p be the perpendicular from C on AB, prove that

$$\sin^2 p\sin^2 c=\sin^2 a\sin^2 b=\sin^2 a+\sin^2 b-\sin^2 c.$$

21. In a quadrantal triangle, c being the quadrant, prove that

$$\tan a\tan b+\sec C=0.$$

22. Solve

(1) the spherical triangle whose sides are 60°, 50°, 40°,

(2) the plane triangle obtained by connecting by straight lines the vertices of this spherical triangle, the sphere on which it is drawn being 2 feet in diameter.

23. Two angles of a spherical triangle are 130° and 145°; show that the third angle must be greater than 95° but less than 165°.

24. In a spherical triangle prove that

$$\cos a=\cos b\cos c+\sin b\sin c\cos A,$$

and show that, if in two triangles $b=b'$, $c=c'$ but $A>A'$, then $a>a'$.

25. Prove that the perpendicular from the vertex of a spherical triangle upon the opposite side divides the angle and that side into parts whose tangents have the same ratio.

26. ABCD is a spherical quadrilateral each of whose sides is an arc of a great circle. If AB=40°, AD=60°, CD=50°, B=C=90°, find BC.

27. In the following cases ABC is a three-sided spherical figure each of whose sides is an arc of a great circle. Select those which are spherical triangles and give your reasons for so doing. Point out why those you reject cannot be spherical triangles.

(1) A= 80°, B=110°, C=160°,
(2) A= 98°, B= 84°, C=160°,
(3) A=100°, B= 50°, C=140°,
(4) A= 67°, B= 58°, C= 80°,
(5) A=130°, B=165°, C=140°,
(6) A= 12°, B= 18°, C= 80°.

28. In a right-angled spherical triangle prove

$$\cos a = \cos b \cos c \quad (1), \qquad \sin b = \sin B \sin a \quad (3),$$
$$\cos B = \cot a \tan c \quad (2), \qquad \sin c = \cot B \tan b \quad (4),$$

and write down the three equations which may be inferred involving the angle A.

29. In a right-angled triangle prove

$$\cos a = \cot B \cot C, \ \cos B = \sin C \cos b, \ \cos C = \sin B \cos c.$$

30. In a quadrantal triangle prove

$$\cos a = -\cot a \cot c \quad (1), \qquad \sin C = \cot a \tan A \quad (3),$$
$$\cos a = -\tan C \cot B \quad (2), \qquad \sin C = \sin B \sin c \quad (4).$$

31. Show that in any spherical triangle

$$\cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B},$$

and that if the three angles of a spherical triangle are together equal to four right angles

$$\cos^2 \frac{c}{2} = \cot A \cot B.$$

32. Show that it is impossible for one side only of a right-angled spherical triangle to be greater than a quadrant.

33. If x is the side of a spherical triangle DEF formed by joining the middle points of the equilateral spherical triangle ABC, of side a, show that x may be determined by the equation

$$2 \sin \frac{x}{2} = \tan \frac{a}{2}.$$

34. In a spherical triangle ABC, the angle C is equal to the sum of the other two angles. Show that the arc joining C to the middle point of AB is equal to one half of AB.

35. Two places are both in latitude 45°, and the difference of their longitudes is 60°; find the distance between them

(i) along the parallel of latitude,

(ii) along a straight line.

The radius of earth is to be taken as 4000 miles.

ANSWERS

I

1. $A = 97° 53' 0''$, $B = 67° 59' 39''$, $C = 84° 46' 40''$.
2. $A = 96° 46' 30''$, $B = 84° 30' 20''$, $C = 125° 28' 13''$.
3. $A = 139° 58' 55''$, $B = 141° 28' 57''$, $C = 109° 57' 57''$
4. $A = 77° 38' 18''$, $B = 101° 42' 58''$, $C = 53° 14' 0''$.
5. $A = B = 147° 36' 3''$, $C = 145° 40' 49''$.
6. $A = 91° 55' 57''$, $B = 63° 43' 41''$, $C = 76° 12'$.
7. $A = 93° 15'$, $B = 105° 25'$, $C = 116° 20'$.
8. $A = 126° 41' 40''$, $B = 143° 54' 25''$, $C = 129° 35' 3''$.
9. $A = 137° 44' 40''$, $B = 146° 41' 12''$, $C = 119° 27' 54''$.
10. $A = 85° 25' 0''$, $B = 103° 19' 12''$, $C = 49° 14' 0''$.

II

1. $A = 83° 35' 30''$, $B = 111° 32' 34''$, $c = 95° 13' 20''$.
2. $A = 133° 36' 0''$, $B = 112° 46' 0''$, $c = 80° 10' 50''$.
3. $A = 110° 48' 12''$, $B = 123° 21' 30''$, $c = 82° 40' 56''$.
4. $A = 122° 43' 0''$, $B = 159° 21' 0''$, $c = 24° 28' 23''$.
5. $A = 116° 6' 0''$, $B = 132° 42' 0''$, $c = 116° 47' 4''$.
6. $B = 109° 50' 50''$, $C = 104° 4' 51''$, $a = 111° 30' 0''$.
7. $A = 71° 46' 0''$, $C = 123° 23' 0''$, $b = 122° 13' 4''$.
8. $A = 139° 58' 55''$, $B = 141° 28' 57''$, $c = 49° 56' 10''$.
9. $C = 75° 46' 33''$, $B = 121° 41' 24''$, $a = 116° 11' 25''$.
10. $A = 91° 36' 0''$, $C = 59° 49' 8''$, $b = 130° 32' 9''$.

III

1. $a = 49° 54' 38''$, $b = 147° 33' 54''$, $c = 143° 14' 34''$.
2. $a = 115° 11' 6''$, $b = 139° 36' 44''$, $c = 59° 13' 30''$.
3. $a = 95° 49' 42''$, $b = 151° 11' 42''$, $c = 105° 38' 0''$.
4. $a = 133° 46' 30''$, $b = 113° 9' 40''$, $c = 80° 10' 50''$.
5. $a = 106° 48' 42''$, $b = 118° 41' 48''$, $c = 119° 17' 13''$.
6. $a = 118° 30' 30''$, $b = 155° 29' 30''$, $c = 81° 11' 2''$.
7. $a = 113° 52' 40''$, $b = 127° 9' 40''$, $c = 90°$.
8. $a = 85° 7' 20''$, $b = 121° 3' 50''$, $c = 129° 9' 7''$.
9. $a = 82° 7' 0''$, $b = 112° 0' 21''$, $c = 95° 13' 20''$.
10. $a = 111° 20' 0''$, $b = 123° 40' 0''$, $c = 82° 40' 56''$.

IV

1. $B = 138° 20' 35''$, $a = 121° 47' 16''$, $c = 75° 4' 15''$.
2. $A = 94° 44' 0''$, $b = 116° 16' 19''$, $c = 103° 48' 0''$.
3. $a = 93° 16' 0''$, $b = 121° 28' 0''$, $C = 100° 20' 0''$.
4. $B = 129° 49' 30''$, $a = 58° 23' 40''$, $c = 145° 44' 57''$.
5. $C = 119° 55' 6''$, $a = 50° 54' 32''$, $b = 37° 47' 18''$.
6. $B = 42° 15' 13''$, $a = 76° 35' 36''$, $c = 40° 0' 10''$.
7. $C = 141° 13' 50'$, $a = 69° 8' 44''$, $b = 131° 3' 56''$.
8. $A = 117° 4' 0''$, $b = 134° 57' 50''$, $c = 104° 42' 0''$.
9. $C = 106° 29' 32''$, $a = 85° 43' 43''$, $b = 123° 4' 21''$.
10. $C = 125° 56' 42''$, $a = 116° 20' 0''$, $b = 92° 50'$.

V

1. $B = 143° 54' 25''$, $c = 109° 4' 25''$, $C = 129° 35' 3''$.
2. $B = 138° 7' 25''$, $c = 119° 17' 13''$, $C = 138° 24' 56''$.
3. $C = 75° 46' 0''$, $B = 46° 7' 15''$, $b = 46° 10' 0''$.
4. $C = 104° 48' 4''$, $B = 41° 38' 15''$, $c = 75° 54' 0''$.
5. $B = 58° 32' 0''$, $A = 86° 44' 0''$, $a = 81° 46' 7''$.
6. $C = 56° 12' 0''$, $A = 107° 36' 9''$, $a = 72° 44' 0''$.
7. $C = 63° 33' 42''$, $A = 94° 3' 50''$, $a = 86° 45' 0''$.
8. $B = 146° 41' 12''$, $A = 119° 27' 54''$, $a = 69° 56' 45''$.
9. $B = 124° 34' 57''$ or $55° 25' 2''$.
   $c = 60° 37' 32''$ or $15° 18' 5''$.
   $C = 98° 32' 34''$ or $17° 25' 33''$.
10. $B = 98° 44' 45''$ or $81° 15' 15''$.
    $a = 60° 16' 12''$ or $37° 35' 1''$.
    $A = 69° 49' 9''$ or $41° 14' 34''$.
11. $B = 49° 25' 12''$, $C = 83° 12' 0''$, $c = 61° 54' 6''$.
12. $B = 45° 2' 10''$, $C = 82° 24' 33''$, $c = 62° 56' 0''$.

VI

1. $b = 61° 20' 0''$, $A = 105° 2' 14''$, $a = 101° 30' 0''$.
2. $c = 58° 18' 36''$, $B = 116° 30' 28''$, $b = 104° 13' 27''$.
3. $c = 46° 30' 0''$, $A = 94° 52' 40''$, $a = 68° 45' 0''$.
4. $c = 108° 39' 11''$, $B = 40° 23' 16''$, $b = 45° 36' 20''$.
5. $a = 51° 6' 12''$, $B = 32° 26' 6''$, $b = 44° 13' 45''$.
6. $b = 107° 19' 52''$, $C = 116° 15' 0''$, $c = 115° 28' 13''$.
7. $b = 115° 33' 56''$ or $64° 26' 4''$.
   $c = 102° 38' 25''$ or $168° 48' 19''$.
   $C = 127° 24' 42''$, or $170° 54' 26''$.
8. $b = 117° 35' 35''$ or $62° 24' 25''$.
   $c = 24° 16' 49''$ or $120° 53' 49''$.
   $C = 26° 50' 24''$ or $109° 34' 34''$.
9. $b = 120° 35' 44''$ or $59° 24' 16''$.
   $c = 64° 20' 4''$ or $153° 0' 5''$.
   $C = 82° 26' 41''$ or $150° 2' 49''$.

10. $b=60^\circ\ 53'\ 2''$.
$C=137^\circ\ 49'$.
$c=109^\circ\ 12'$.
11. $b=124^\circ\ 20'\ 3''$ or $55^\circ\ 39'\ 57''$.
$c=63^\circ\ 25'\ 42''$ or $155^\circ\ 27'\ 45''$.
$C=86^\circ\ 51'\ 50''$ or $152^\circ\ 22'\ 40''$.
12. $b=125^\circ\ 53'\ 41''$ or $54^\circ\ 6'\ 19''$.
$a=63^\circ\ 4'\ 34''$ or $155^\circ\ 47'\ 7''$.
$A=89^\circ\ 28'\ 14''$ or $152^\circ\ 36'\ 46''$.

VII

1. $B=47^\circ\ 44'\ 15''$, $A=54^\circ\ 47'\ 54''$, $c=50^\circ\ 7'\ 13''$.
2. $c=66^\circ\ 31'\ 58''$, $a=37^\circ\ 48'\ 22''$, $b=59^\circ\ 44'\ 3''$.
3. $c=70^\circ\ 19'\ 40''$, $A=52^\circ\ 42'\ 9''$, $B=66^\circ\ 10'\ 29''$.
4. $c=55^\circ\ 20'\ 23''$, $A=51^\circ\ 3'\ 4''$, $B=63^\circ\ 45'\ 0''$.
5. $a=73^\circ\ 19'\ 28''$, $B=70^\circ\ 25'\ 33''$, $c=48^\circ\ 12,\ 0''$.
6. $a=85^\circ\ 19'\ 2''$, $B=87^\circ\ 21'\ 0''$, $C=29^\circ\ 33'\ 0''$.
7. $A=125^\circ\ 12'\ 6''$, $B=47^\circ\ 44'\ 0''$, $b=34^\circ\ 36'\ 11''$.
8. $b=47^\circ\ 57'\ 15''$, $a=156^\circ\ 10'\ 34''$, $C=113^\circ\ 28'\ 0''$.
9. $a=61^\circ\ 4'\ 56''$, $B=61^\circ\ 50'\ 28''$, $C=47^\circ\ 54'\ 21''$.
10. $a=131^\circ\ 30'\ 0''$, $b=59^\circ\ 28'\ 0''$, $B=66^\circ\ 9'\ 26''$.
11. $A=127^\circ\ 19'\ 13''$, $b=120^\circ\ 32'\ 0''$, $a=131^\circ\ 30'\ 0''$.
12. $a=56^\circ\ 7'\ 29''$ or $123^\circ\ 52'\ 30''$.
$b=42^\circ\ 37'\ 43''$ or $137^\circ\ 22'\ 17''$.
$B=54^\circ\ 39'\ 26''$ or $125^\circ\ 20'\ 34''$.
13. $a=70^\circ\ 23'\ 42''$ or $109^\circ\ 36'\ 18''$.
$b=48^\circ\ 39'\ 16''$ or $131^\circ\ 20'\ 44''$.
$B=52^\circ\ 50'\ 20''$ or $127^\circ\ 9'\ 40''$.
14. $c=41^\circ\ 12'\ 54''$ or $138^\circ\ 47'\ 5''$.
$b=61^\circ\ 57'\ 58''$ or $118^\circ\ 2'\ 1\frac{1}{2}''$.
$C=48^\circ\ 17'\ 8''$ or $131^\circ\ 42'\ 52''$.

VIII

1. $a=125^\circ\ 12'\ 6''$, $B=145^\circ\ 23'\ 49''$, $C=129^\circ\ 52'\ 47''$.
2. $a=128^\circ\ 56'\ 56''$, $B=124^\circ\ 39'\ 37''$, $C=113^\circ\ 29'\ 29''$.
3. $c=109^\circ\ 34'\ 26''$, $b=128^\circ\ 54'\ 13''$, $A=106^\circ\ 40'\ 31''$.
4. $A=150^\circ\ 33'\ 30''$, $C=95^\circ\ 22'\ 45''$, $B=94^\circ\ 40'\ 58''$.
5. $A=123^\circ\ 52'\ 30''$ or $56^\circ\ 7'\ 30''$.
$B=137^\circ\ 22'\ 17''$ or $42^\circ\ 37'\ 43''$.
$b=125^\circ\ 20'\ 34''$ or $54^\circ\ 39'\ 26''$.
6. $A=125^\circ\ 18'\ 25''$, $b=78^\circ\ 12'\ 4''$, $B=75^\circ\ 38'\ 32''$.
7. $B=129^\circ\ 29'\ 31''$, $b=118^\circ\ 9'\ 32''$, $a=132^\circ\ 5'\ 39''$.
8. $B=138^\circ\ 47'\ 5''$ or $41^\circ\ 12'\ 55''$.
$A=118^\circ\ 2'\ 1''$ or $61^\circ\ 57'\ 59''$.
$b=131^\circ\ 42'\ 52''$ or $48^\circ\ 17'\ 8''$.
9. $a=127^\circ\ 47'\ 23''$, $c=114^\circ\ 18'\ 23''$, $C=121^\circ\ 23'\ 22''$.

10. $A=164° 3' 30''$, $C=133° 17' 48''$, $c=104° 31' 25''$.
11. $A=135° 6' 51''$, $B=110° 27' 5''$, $b=104° 44' 38''$.
12. $A=138° 42' 12''$, $a=127° 4' 12''$, $C=116° 55' 43''$.
13. $a=93° 5' 7''$, $b=71° 41' 39''$, $C=78° 58' 41''$.
14. $A=111° 17' 1''$ or $68° 42' 59''$.
$B=119° 23' 50''$ or $60° 36' 10''$.
$b=110° 46' 13''$ or $69° 13' 47''$.

IX

1. $a=69° 8' 44''$, $B=130° 35' 50''$, $C=141° 13' 50''$.
2. $A=111° 39' 35''$, $B=127° 41' 45''$, $C=83° 39' 16''$.
3. $b=122° 25' 9''$ or $57° 34' 51''$.
$C=84° 41' 44''$ or $151° 14' 55''$.
$c=64° 2' 9''$ or $154° 15' 28''$.
4. $a=138° 4' 15''$, $B=120° 15' 44''$, $C=113° 27' 54''$.
5. $a=112° 23' 0''$, $b=140° 41' 0''$, $c=76° 2' 44''$.
6. $A=126° 35' 48''$, $B=142° 32' 5''$, $c=76° 47' 33''$.
7. $a=37° 48' 12''$, $b=59° 44' 16''$, $c=66° 32' 6''$.
8. $A=86° 54' 53''$, $b=108° 20' 0''$, $c=91° 1' 18''$.
9. $A=52° 12' 37''$, $b=58° 36' 38''$, $c=69° 30' 0''$.
10. $a=75° 12' 0''$, $b=63° 27' 0''$, $c=52° 23' 0''$.
11. $a=109° 37' 44''$, $b=130° 34' 48''$, $C=118° 5' 54''$.
12. $a=125° 20' 0''$, $B=135° 36' 20''$, $C=139° 59' 56''$.
13. $A=113° 40' 16''$, $B=134° 53' 27''$, $C=103° 32' 0''$.
14. $b=116° 59' 35''$, $C=105° 20' 0''$, $c=105° 53' 39''$.
15. $a=88° 35' 41''$, $b=112° 20' 14''$, $c=125° 0' 38''$.
16. $a=84° 3' 42''$, $B=133° 50' 0''$, $C=104° 3' 33''$.
17. $a=72° 23' 57''$, $b=143° 42' 39''$, $C=123° 38' 30''$.
18. $c=63° 4' 17''$ or $116° 55' 43''$.
$b=52° 55' 47''$ or $127° 4' 13''$.
$B=63° 30' 0''$ or $116° 30' 0''$.
19. $c=48° 45' 0''$ or $131° 15' 0''$.
$b=46° 42' 12''$ or $133° 17' 48''$.
$B=75° 28' 35''$ or $104° 31' 25''$.
20. $b=104° 44' 38''$ or $75° 15' 22''$.
$C=104° 20' 0''$ or $75° 40' 0''$.
$B=110° 27' 5''$ or $69° 32' 55''$.

X

(1) 35 inches. (2) 3 inches. (3) $2\sqrt{17}$ inches. (4) 12 inches, $6\frac{2}{7}$ inches. (5) 3 : 5. (6) 14 inches, 7 inches. (7) $4\sqrt{17}$ inches. (8) $46° 23' 45''$. (9) 418 miles. (10) 5 : 8 : 10. (11) 800 miles. (12) $3° 2' 42''$. (13) $1 : \sqrt{2} : \sqrt{3} : 2$. (14) 50° N., 70° E. (16) $1 : \sqrt{2}$.

XI

(1) $37° 12' 45''$. (2) $42° 20' 13''$; $114° 35' 45''$; $31° 40'$. (3) $113° 16' 30''$. (4) $99° 46' 15''$. (5) $70° 31' 45''$. (6) $109° 28' 15$; $54° 44'$. (7) (*a*) $51° 19'$, (*b*) $64° 20' 30''$, (*c*) $67° 22' 45''$, (*d*) $73° 53' 45''$. (10) 60°; $102° 39' 45''$. (12) $a=49' 0''$; length $=57$ miles. (13) A must lie between 45° and 135°.

XII

(4) $\cos a = \cos b \cos c$; $\cos A = -\cot b \cot c$; $\cos a = \dfrac{\cos A + \cos B \cos C}{\sin B \sin C}$. (5) $133^\circ\ 28'\ 30''$. (7) each angle $= 70^\circ\ 31^\circ\ 45''$. (8) $\sin \dfrac{x}{2} = \dfrac{\sec \frac{A}{2}}{2}$, where A is the angle contained by the semicircles, and x is the arc taken on one of them. It is impossible when $\sec \dfrac{A}{2} = 2$ or > 2. (9) $\dfrac{1}{3}$. (10) $\dfrac{203}{480}$. (19) cot course $= \dfrac{\sqrt{2}}{2}$, distance $= 3600$ nautical miles.

(22) (1) $A = 47^\circ\ 54'\ 45''$, $B = 89^\circ\ 7'\ 0''$, $C = 62^\circ\ 11'\ 0''$,

(2) $A = 42^\circ\ 29'\ 15''$, $B = 80^\circ\ 56'\ 0''$, $C = 56^\circ\ 34'\ 45''$.

(26) $89^\circ\ 7'$. (27) (1) no, (2) yes, (3) no, (4) yes, (5) yes, (6) no. (35) (i) 3381, (ii) 2963.

THE END

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